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Subsidizing Research programs with "If" and "When" Uncertainty in the Face of Severe Information Constraints

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Subsidizing Research Programs with "If" and "When" Uncertainty in the Face of Severe Informational Constraints*

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Abstract

We study government optimal subsidy policies for research programs in the face of servere information asymmetry—when firms have private information about the likelihood of project viability but the government cannot form a unique prior belief about this likelihood. The paper makes two contributions. First, we show that the *way* in which R&D is subsidized matters. Under both monopoly R&D (i.e., a single firm conducts R&D in isolation) and R&D competition, different types of subsidies (e.g., earmarked, unrestricted subsidies, and pure matching subsidies) have significantly different effects on firms' R&D investment incentives. Second, we show that a *simple* subsidy scheme works even when the government is unable to form a unique prior belief about the firm's private information on project viability. If the shadow cost of public funds is zero, under monopoly R&D, there exists a pure matching subsidy that induces the firm to follow the first-best R&D policy irrespective of its prior beliefs about the viability of the project, meaning it is a (belief-free) *ex post* equilibrium policy; under R&D competition, the first-best outcome can also be achieved through a simple combination of a matching subsidy and an unrestricted subsidy. If the shadow cost of public funds is positive, an *ex post* equilibrium in general does not exist either under monopoly or competition. We then consider two alternative policy decision criteria that are appropriate for belief-free games: rationalizability and max-min criteria. We argue that the max-min criteria is preferable in our context, and by way of doing so establish that the set of max-min subsidy policies under either monopoly or competitive R&D consists entirely of simple pure matching subsidies. We further establish that allowing firms to form an R&D consortium reduces the matching rate for the highest max-min subsidy, suggesting that cooperative R&D has the potential to economize on the shadow costs of public funding of subsidies.

Keywords: Research and development, subsidies, dyanmic stochastic game, aymmetric information, belief-free game social cost of public funds, research consortia.

JEL classification: O38, D60, D82, H2, C73.

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1 Introduction

Governments have long played a role in subsidizing private investments in R&D. The principal economic justification for R&D subsidies is the presence of market imperfections (e.g., limits on appropriability or problems of free riding) that result in socially suboptimal provision of private R&D (Arrow, 1962). But the economic literature is not clear how successful subsidies have been, or can be, in addressing these market failures. The empirical literature on R&D subsidies presents decidedly mixed results, with some studies concluding that R&D subsidies stimulate private R&D investment (e.g., Lach, 2002 and Almus and Czarnitzki, 2003), while others finding that subsidies crowd out private R&D investment (e.g., Irwin and Klenow, 1996 and Wallsten, 2000) or leave it unchanged (e.g., Klette and Moen, 1998, 1999). The theoretical literature (discussed in more detail below) shows that subsidies can, in principle, stimulate private R&D investment and increase social welfare, but the models in the literature are typically cast within static and/or deterministic settings that seem far removed from the dynamic and uncertain environments in which much modern research (especially basic research) takes place. It is fair to say that from both a theoretical and empirical perspective, the effectiveness of government R&D subsidies is an unsettled question, and as Hall (2005) suggests, it deserves further research.

The purpose of this paper is to advance the theory of R&D subsidies by studying their impact in a setting with two key features: (1) firms receiving subsidies for a research program are uncertain both about the *timing* of the scientific breakthrough that the program can lead to ("when uncertainty") and about the underlying *viability* of the program itself ("if uncertainty"); (2) firms' prior beliefs about program viability are private information, and the government providing the subsidies is unable to form a unique prior belief about the firms' priors. That is, the subsidizing government faces a severe informational asymmetry—it neither knows how optimistic firms are about the viability of the research program, nor does it know what to believe about firms' optimism. These features seem especially likely to hold for research programs taking place in "uncharted waters," so our theory is particularly relevant for groundbreaking research programs in areas in which there is little established consensus about whether the current direction of inquiry is likely to be fruitful.

Our paper makes two broad contributions. First, we show that the *way* in which R&D is subsidized matters. Certain types of subsidies (i.e., earmarked funding in which a firm is required to spend a certain minimum amount on the project) may crowd out private investment, while other types of subsidies (i.e., a pure matching subsidy in which a firm undertaking R&D is reimbursed a fraction of its R&D expenses) may stimulate private investment. This suggests that empirical studies of the impact of subsidies on private R&D investment need to be cognizant of how subsidies are structured. Second, we show that *simple* R&D subsidy scheme works. Despite the government's severe informational constraints, simple subsidy mechanisms—in particular a pure matching subsidy, in which the government reimburses a fraction of the firms' R&D expenditures—can still perform reasonably well with respect to plausible decision criteria, and under interesting circumstances it can even attain the first-best level of welfare.

More specifically, our paper uses a two-armed bandit model of R&D competition in which firms seek to achieve a significant scientific breakthrough.[1] As time passes and the breakthrough is not achieved, firms become more and more pessimistic about the likelihood that this path of inquiry will ever pay off, and if they become sufficiently pessimistic, they will eventually terminate the project. Conditional on the project being viable, the likelihood and timing of a breakthrough depends on how persistent the firms are, i.e., how willing they are to continue to fund the project over time. To incentivize firms' private R&D investment, the government implements a subsidy mechanism in which a firm's R&D subsidy is a function of its actual R&D effort. The mechanism subsumes the three specific funding schemes commonly used in practice: a pure matching subsidy, an earmarked subsidy, and an unrestricted subsidy in which the government makes an open-ended commitment to fund the project until a breakthrough occurs (though unlike an earmarked subsidy there is no formal requirement that the firm actually spend the money on the focal R&D project). In other words, the government's subsidy scheme has three components—a combination of matching, earmarked and unrestricted subsidies.

We first study the impact of this subsidy mechanism under both monopoly R&D and R&D competition. The focus here is to identify the incentive effects of subsidizing R&D when there is both "if" and "when" uncertainty. Under a monopoly R&D, the firm's optimal R&D investment decision is a "bang-bang" rule: depending on its posterior beliefs about the project's viability, it either invests "flat out" in R&D at each instant in time or not at all. Compared with the case of no subsidies, the matching component of the subsidy expands the range of posterior beliefs over which the monopolist invests "flat out," while the earmarked and unrestricted components of the subsidy shrink that range. Thus, increases in

[1] Using two-armed bandit model to analyze economic problems dates back to Rothschild (1974). Recently, a number of papers focus on the strategic interaction among agents in a bandit framework (e.g., Keller, Rady, and Cripps, 2005, and Klein and Rady 2008). Our paper is closest to Keller, Rady, and Cripps (2005), as our second stage R&D competition is based on their Poisson bandit framework. Besanko and Wu (2013) explore R&D competition and cooperation in a model inspired by Keller, Rady, and Cripps (2005).

the matching rate stimulate private spending on R&D, while increases in unrestricted funding or the minimum mandated R&D effort crowd out private spending. But a subsidy with minimum mandated R&D effort could also increase R&D investment. Under an earmarked subsidy with minimum mandate, R&D effort on the project continues below the point at which the firm stops investing without subsidy, and thus lead to more overall investment in R&D. Under R&D competition, an additional complication arises that does not exist under monopoly: the possibility that firms may free ride on the R&D efforts of other firms. As we show below, the free-rider problem implies that in a symmetric equilibrium, investment by firms is no longer "bang-bang." Instead, it may involve a range of posterior beliefs over which each firm invests a positive amount in R&D which is less than the technological maximum. When the free-rider problem can arise, an unrestricted component to the subsidy and a minimum mandated level of R&D—which had unambiguously adverse incentive effects under monopoly—can eliminate the free-rider problem. Indeed, without the unrestricted component or a minimum mandate, the free-rider problem always arises.

We then turn to optimal subsidy policy in a setting in which firms have private information about the prior likelihood that the project is viable, and the government lacks probabilistic knowledge of this likelihood; in other words, the government has no (unambiguous) prior over the firm's prior. This severely constrained informational environment prevents the government from using either a forcing mechanism to achieve first-best welfare or a conventional mechanism design approach in which it offers the firms a menu of policies that maximizes expected social welfare for a given prior probabilistic belief over the firms' private information. Under monopoly R&D, we show that when a subsidy is a pure transfer between taxpayers and firms, meaning no shadow cost of public funding, there exists a pure matching subsidy that induces the firm to follow the first-best R&D policy irrespective of its prior beliefs about the viability of the project. This particular matching subsidy is thus a (belief-free) *ex post* equilibrium. To implement it, the government only needs to estimate the total social benefit and the percentage of which that can be appropriated by the firm. By contrast, when there is a positive shadow cost of public funds, we prove that an *ex post* equilibrium does not, in general, exist. We then consider two alternative policy making criteria that are appropriate for belief-free games[2] of the sort we consider here: rationalizability (Pearce, 1984) and the max-min criteria (Gilboa and Schmeidler, 1989). Through its implication, rationalizability prevents the policy maker from choosing strictly dominated

[2]See Bergemann and Morris (2007) for a general treatment of belief-free incomplete information games.

policies, while max-min policies protect the policy maker against inferior "worst case scenario" policy outcome relative to any alternative policies. Both criteria strike us as plausible inasmuch as policy makers operating under significant informational constraints may very well be keen to avoid big policy "mistakes." We show that policies satisfying the max-min criteria are rationalizable, but the converse is not true. We further show that the set of (pure strategy) max-min policies are pure matching subsidies, and the policy in this set with the highest matching rate has the appealing property of eliminating the possibilities of both underinvestment and overinvestment in R&D. Further, as the shadow cost of public funds goes to 0, this matching rate approaches the matching rate that implements the first-best solution. For these reasons, we suggest that the max-min criterion is preferable to rationalizability in our context.

The first-best outcome can also be achieved under R&D competition when there is zero shadow cost of public funds. However, unlike monopoly, that policy is not a pure matching policy. Instead, it involves a combination of a matching subsidy and an unrestricted subsidy. The unrestricted component of the subsidy eliminates the free-rider problem, and given this, the matching rate is then set to mimic the social planner's optimal investment policy. Like the monopoly case, when the shadow cost of public funds is positive, an *ex post* equilibrium does not exist. Policies that satisfy the max-min criterion are again pure matching subsidies. But unlike the monopoly case, the policy in the max-min set with the highest matching rate cannot overcome suboptimal intensity of investment due to the free-rider problem. However, if the government permits firms to choose R&D cooperatively through a research consortium, we show that there exists a matching rate that satisfies the max-min criterion and eliminates both over- and underinvestment.

Our paper fits within the theoretical literature on R&D subsidy policy. Papers in this literature have focused on a number of broad issues. Some, such as Spencer and Brander (1983) and Qiu and Tao (1998), study the use of R&D subsidies to enhance national competitiveness. Other papers consider the role of subsidies to help overcome informational problems. For example, Socorro (2007) explores optimal patent subsidies for R&D in the context of a mechanism design problem in which firms have private information about the value of an uncertain R&D project, while Takalo and Tanayama (2010) examine whether R&D subsidies can alleviate financing constraints due to adverse selection. Most closely related to this paper are papers by Hinloopen (1997, 2000), Stenbacka and Tomback (1998), and Romano (1989) that explore the impact of subsidies on the level of R&D and social welfare. Hinloopen

(1997) analyzes a model similar to the framework of d'Aspremont and Jacquemin (1988) and Kamien et. al. (1992) in which firms' investments in cost-reducing effort deterministically reduces their costs, and possibly the costs of other firms as well due to spillovers. R&D subsidies are shown to increase the level of investment activity and social welfare and are more effective at increasing R&D investment than allowing firms to cooperate through research joint ventures or R&D cartels. Stenbacka and Tomback (1998) analyze the best way to organize R&D (e.g., competitive research joint venture, cartelized research joint venture, R&D competition) given that the government chooses an optimal subsidy rate for the mode of organization being considered. With optimal subsidies, research joint ventures are shown to be superior to competition provided that the social cost of subsidies is not too large. Romano (1989) analyzes subsidies for research projects aimed at achieving process innovations in the presence of "when" uncertainty. He shows that it is always socially optimal to subsidize a monopolist, but under certain circumstances (e.g., sufficiently long patent life) it is not optimal to subsidize competitive firms.

Our paper differs from the existing theoretical literature in several important respects. First, unlike the existing literature that analyze R&D subsidies in reduced-form static or two-stage models, our model is explicitly dynamic. By employing the two-armed bandit framework, we can analyze how alternative subsidy policies affect belief updating and the abandonment of R&D projects, issues that cannot be studied in static or two-stage models. Second, we consider more general subsidy policies than those considered in the existing literature. Hinloopen (1997, 2000) and Stenbacka and Tomback (1998) consider pure matching subsidies, while Romano considers unrestricted subsidies. Our paper, by contrast, analyzes a more general subsidy mechanism that embraces both matching and unrestricted subsidies as special cases. Third, in contrast to many of the papers cited above, a key focus of our paper is on the properties of an optimal subsidy policy and how that policy is affected by underlying economic fundamentals. Finally, we consider an environment in which the policy maker lacks prior beliefs over firms' private information about project viability. Accordingly, optimal subsidy policy in our model cannot depend on the details of potentially *ad hoc* subjective beliefs and must instead be robust to the entire range of possible assessments that the firms might have about project viability.

Our paper is also related to several papers in the broader literature on the financing of innovative activity, in particular Bergemann and Hege (1998, 2005) and Hörner and Samuelson (2013). These papers, like ours, study R&D projects that are characterized by both "if" and "when" uncertainty. The main focus of these papers is to explore the hidden action (and its induced hidden information) problems in a context of open-ended exclusively external financing. Although it is critical to understand the problem caused by hidden action in R&D experimentation, it is equally important to understand the economics of R&D subsidies in the case of severe information asymmetries. By shifting the emphasis from hidden action to the hidden information problem in the context of a belief-free incomplete information game, our paper complements the current literature on funding experimentation by including a number of new important features. First, unlike exclusively external funding, we allow firms to use their own funding to pursue R&D while receiving financial support from the government. Second, we address the appropriability and free-riding problems simultaneously with a funding scheme that includes matching, earmarked, and unrestricted subsidies as special cases, while the current literature largely restricts the external financial support to unrestricted funding only. Third, we study the funding policy for multiple firms while the Bergemann and Hege and Hörner and Samuelson papers focus on the one-firm case. Finally, their models assume no friction in external funding, while we consider cases with and without frictions in terms of a shadow cost.

The paper is divided into five sections, including this one. Section 2 describes the model. Section 3 illustrates how changes in (given and possibly non-optimal) subsidy policies affects the firm's R&D incentives, first for the case of monopoly R&D and then for the case of R&D competition. Section 4 then takes up the question of how the government, lacking both deterministic and probabilistic knowledge of firms' priors about the viability of the project, would determine an optimal R&D subsidy policy. Again, we start with the case of monopoly R&D and then turn to R&D competition. Section 5 summarizes and concludes. Proofs of all propositions are in the Appendix.

2 The Model

We present a model of R&D investment based on the exponential bandit framework of Keller, Rady, and Cripps (2005). We state the model with N firms, with the analysis of monopoly corresponding to the special case of $N = 1$.

Each of the N identical firms faces an opportunity to invest in an R&D program aimed at achieving a significant breakthrough. *Ex ante* the firms do not know if a breakthrough is possible. Let p_0 denote the firms' common prior that the project is viable, i.e., that the breakthrough can be achieved eventually. Conditional on the project being viable, the time the breakthrough occurs is random.

Higher R&D investment increases the likelihood that the breakthrough occurs sooner. Specifically, let k_t^i, $i = 1, \ldots N$, denote firm i's R&D investment at time t. Conditional on the project being viable, the hazard rate of firm i's success on the R&D project is $\lambda k_t^i dt$, where $\lambda > 0$ is a parameter. We assume that each firm faces a technological constraint that limits its investment in R&D to at most 1 unit of effort at any point in time t, meaning $k_t^i \in [0, 1]$. One can interpret this constraint as an extreme form of diminishing marginal returns to R&D. If the project was indeed viable, and a single firm exerted the maximum feasible level of R&D effort ($k = 1$), then $\frac{1}{\lambda}$ would be the expected time until a breakthrough occurs. R&D effort is costly, and the total cost $C(k_t^i)$ of R&D effort is assumed to be an identical linear function for each firm, $C\left(k_t^i\right) = \alpha k_t^i$, where $\alpha > 0$ denotes the marginal cost of R&D effort.[3]

The achievement of a breakthrough is assumed to be "big news" and visible to all firms competing in the R&D race. As time passes and a breakthrough has not occurred, firms become more pessimistic about the viability of the project. Let $p\left(t\right)$ denote firms' posterior belief about the project's viability at date t. If no breakthrough occurs, $p(t)$ adjusts downward according to Bayes rule:

$$p\left(t + dt\right) = \frac{p\left(t\right)\left(1 - \lambda \sum_{i=1}^{N} k_t^i dt\right)}{1 - p\left(t\right) + p\left(t\right)\left(1 - \lambda \sum_{i=1}^{N} k_t^i dt\right)}. \tag{1}$$

It follows that

$$\frac{dp}{dt} = \lim_{dt \to 0} \frac{p\left(t + dt\right) - p\left(t\right)}{dt} = -\lambda \sum_{i=1}^{N} k_t^i p\left(t\right)\left(1 - p\left(t\right)\right). \tag{2}$$

This rate of belief updating is independent of its starting state, so we may rewrite it as

$$dp = -\lambda \sum_{i=1}^{N} k_t^i p\left(1 - p\right) dt. \tag{3}$$

The solution concept is Markov Perfect Equilibrium[4], with each firm's common posterior belief p being the payoff-relevant state variable and equation (3) representing the law of motion for the state variable. Investment behavior and firm value functions are thus conditioned on p. It is straightforward to establish that for the important case of constant "flat out" investment, i.e., $k_\tau^i = 1$ for $\tau \in [0, t]$ and all $i = 1, \ldots N$,

[3] The linearity of the cost function is needed to solve for the equilibrium investment level in closed form. The basic intuition underlying the results does not depend on the linearity of the cost function.

[4] Our use of the term Markov Perfect Equilibrium does not restrict it to be a refinement of subgame perfect Nash equilibrium. This relaxation allows us to apply it to our particular dynamic game with incomplete information. Since the government is the first mover who moves only once, to satisfy the requirement of Markov perfection, we only need each firm's strategy to be Markovian, and solve the appropriate dynamic programming problem.

the posterior belief about project viability if no breakthrough has occurred by date t is given by $p(t) = \frac{p_0}{p_0+(1-p_0)e^{\lambda t}}$.[5] When the prior belief is close to 1, the posterior belief evolves very slowly for a long period of time if no breakthrough occurs, and in the extreme case of $p_0 = 1$, there is no "if" uncertainty, and beliefs remain at $p_0 = 1$ even as time passes without a breakthrough.[6]

We assume the firm that wins the R&D race earns a payoff $\Pi > 0$. This payoff is the present value of the winning firm's profits, which are assumed to be discounted at a rate r. Each of the $N-1$ non-winning firms is assumed to receive a payoff $\theta\Pi$, where $\theta \in [0,1]$. If $\theta = 0$, the R&D race is winner-take-all; if $\theta > 0$, the breakthrough has positive spillovers. The discounted present value of the social benefit from the new technology is given by $CS + \Pi + (N-1)\theta\Pi$, where CS is the benefit of the breakthrough for consumers that the discovering and non-discovering firms cannot capture.[7] For later use, let $\rho \triangleq \frac{\Pi+(N-1)\theta\Pi}{CS+\Pi+(N-1)\theta\Pi} \in (0,1]$ be the appropriability ratio, i.e., the share of the total benefit captured by firms and thus $1-\rho$ is the share of the total benefit that accrues to consumers. Therefore, government intervention in the market for R&D has two potential justifications: the presence of R&D spillovers across firms (when $\theta > 0$) and the imperfect appropriability of the benefit from the breakthrough (when $\rho < 1$). Throughout the analysis, we assume that $\frac{\lambda[CS+\Pi+(N-1)\theta\Pi]}{\alpha} > 1$, which implies that the social benefit-cost ratio of a viable R&D project exceeds 1.[8]

The government's R&D subsidy policy is represented by an array of three instruments, $\mathbf{a} = (z, s, \phi)$, that form a schedule $S(k_t^i|\mathbf{a})$ that determines the funding flow a firm receives at each instant in time prior to a breakthrough:

$$S(k_t^i|\mathbf{a}) = \begin{cases} s + \phi\alpha(k_t^i - z) & \text{if } k_t^i \geq z, \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

In this schedule:

- $z \in [0,1]$ is the minimum R&D effort mandated by the government at each instant in time in

[5] As we will see, constant investment, at least for a while, occurs along the equilibrium path for both $N=1$ and $N>1$.

[6] Suppose for example, if $p_0 = 0.9999$ and $\lambda = 0.1$. If firms are investing "flat out," then by $t = 50$, the posterior is still greater than 0.98.

[7] Throughout the analysis, we assume that Π, CS, and θ do not depend on N. In other words, we assume that the structural conditions that determine post-breakthrough profit, consumer surplus, and spillovers are independent of the number of firms engaged in competition to achieve the breakthrough itself.

[8] Let $\widetilde{T}$ be the random time to discovery for a project that is certain to be viable. With hazard rate λ and flat-out investments (by all N firms) at any point in time, $\widetilde{T}$ is an exponential random variable with parameter $N\lambda$. The *ex ante* expected social benefit of a viable R&D project would be $[CS + \Pi + (N-1)\theta\Pi]\, E(e^{-r\widetilde{T}})$, which equals $\frac{N\lambda[CS+\Pi+(N-1)\theta\Pi]}{r+N\lambda}$. The *ex ante* expected cost of a viable R&D project would be $\int_0^\infty N\alpha \left(\int_0^t e^{-r\tau} d\tau\right) \lambda e^{-N\lambda t} dt$ which can be shown to equal $\frac{N\alpha}{r+N\lambda}$. The *ex ante* benefit cost ratio is thus $\frac{\lambda[CS+\Pi+(N-1)\theta\Pi]}{\alpha}$.

order for the firm to be eligible for any funding, and thus αz is the minimum mandated spending on R&D.

- $s \in [\alpha z, \bar{s}]$ is the baseline amount of funding the firm receives, provided it satisfies the mandate, where $\bar{s}$ is a finite limit of s assumed to be such that $\frac{\bar{s}}{r} < \Pi$.[9] We require that $s \geq \alpha z$ so that at any point in time the firm would prefer to adhere to the mandate and accept the associated funding, rather than reject it. If $s = \alpha z$, the government exactly reimburses the firm its mandated R&D spending, while if $s > \alpha z$, the firm receives a subsidy in excess of its minimum mandated R&D expenditure. In this latter case, the firm could (in principle) spend some of its government funds on other activities besides the focal R&D project (e.g., it could fund other R&D projects). We thus refer to $s - \alpha z$ as the *unrestricted component* of its R&D subsidy.

- $\phi \in [0, 1]$ is the matching rate: the additional funding the firm receives for every additional dollar of R&D spending undertaken above the mandated level.

Throughout we let $A \triangleq \{\mathbf{a} | z \in [0, 1], s \in [\alpha z, \bar{s}], \phi \in [0, 1]\}$ denote the set of feasible subsidy policies. The set of policies in A embrace three interesting special cases:

- If $z = 0$, $s = 0$, and $\phi \in (0, 1]$, then a firm receives a *pure matching subsidy*: for every αk dollars of R&D investment, the government "matches" the firm's R&D spending by providing a subsidy of $\phi \alpha k$.

- If $z > 0$, $s = \alpha z$, and $\phi = 0$, then a firm receives an *earmarked subsidy*: it receives a subsidy of αz dollars, provided that its R&D effort satisfies the mandate z.

- If $z = 0$, $s > 0$, and $\phi = 0$, then a firm receives a *pure unrestricted subsidy*: it receives a no-strings-attached grant of s.

We assume throughout that the experience and scientific facts that underpin p_0 are unknown to the government and are thus private information to the firms. Therefore, p_0 can be interpreted as the firms' unobservable type. Lacking knowledge of p_0, the government cannot infer the firms' posterior belief $p(t)$ and, therefore, cannot write a "forcing contract" in which it ties the parameters of the subsidy function

[9] This implies that an unrestricted subsidy can never be so large that a firm would (weakly) prefer collecting the subsidy to receiving the prize with certainty.

to the posterior in such a way that it replicates the first-best investment policy.[10] We further assume, in contrast to the conventional Bayesian approach in which the government would have a given prior belief about p_0, that the government has no such prior. In other words, the government not only lacks complete information about p_0, but it also lacks probabilistic information about it. Without a prior over p_0, the government cannot determine an optimal menu of policies that maximizes its expected welfare, as in a conventional mechanism design approach. An advantage, in our view, of this assumption is that it forces consideration of robust policies that would not need to be fine-tuned in the wake of changes in potentially arbitrary prior beliefs.

Note that we focus exclusively on *ex ante* subsidies that are paid during the research phase of the project. We do not consider *ex post* subsidies that are contingent on the success of the project (sometimes called patent subsidies; see Socorro, 2007).[11] Further, we restrict attention to subsidy policies that are both time invariant (i.e., (z, s, ϕ) are independent of t) and common to all firms in the industry. Focusing on such policies is not only useful for building intuition about the impact of alternative subsidy instruments on firms' incentives, but as we show below, restricting attention to such policies may entail no loss of generality since, under important circumstances, they are powerful enough to attain the first-best outcome.[12]

3 The Incentive Effects of R&D Subsidies

Why does the *way* in which R&D is subsidized matter? We answer this question by examining the incentive effects of various R&D subsidy schemes, starting with the case of a single firm in isolation (monopoly R&D) followed by the case of multiple firms in an R&D competition. In particular, we derive the optimal R&D strategies given a subsidy policy and show how variations of subsidy policy affect firms' incentives in R&D investment.

[10] The first-best investment policy for $N = 1$ is stated in the next section and in the section after that for $N > 1$.

[11] We focus on *ex ante* subsidies because they are the most common way that governments support private R&D activity. In our model, patent subsidies would enable the government to achieve the first-best solution under monoply when the shadow cost of public funds is zero. Even in this case, though, they would require that the government pass along the entire consumer surplus to the winning firm. This may be possible if the government itself is the only consumer of the products created by the research, as in the case of defense-related R&D. But this could be difficult in the context of other crucial technologies such as stem cell research.

[12] A particular example of a time-varying subsidy policy would be one in which funding is cut off after a certain deadline. Bonatti and Hörner (2011) analyze the role of deadlines in collaborative research, and they show how deadlines can overcome the moral hazard in teams. In the concluding section, we discuss the role that deadlines might play in our model and how our model relates to the insights developed by Bonatti and Hörner.

3.1 Monopoly R&D ($N = 1$)

Faced with a subsidy policy $\mathbf{a}$, the single firm in the case of monopoly R&D has a Bellman equation as follows:

$$V(p) = \max_{k\in[z,1]} \left[(s - \alpha\phi z - \alpha(1-\phi)k)dt + \lambda kpdt\Pi + (1 - \lambda kpdt)\, e^{-rdt} V(p + dp) \right]. \tag{5}$$

With probability $\lambda kpdt$, a breakthrough will take place within the interval $[t, t+dt)$, which gives the firm the prize Π from discovering the new technology. With probability $1 - \lambda kpdt$, no breakthrough occurs within the interval $[t, t+dt)$, and the firm's value become $e^{-rdt}V(p+dp)$. Using standard arguments, the firm's value function can be shown to be implicitly defined by the following differential equation:[13]

$$(1+r)V(p) = s - \alpha z + \max_{k\in[z,1]} \left\{ \begin{array}{c} [\lambda kp\Pi + (1-\lambda kp)V(p)] \\ -\alpha(1-\phi)(k-z) - \lambda kp(1-p)V'(p) \end{array} \right\}. \tag{6}$$

The firm's value consists of four components:

1. $s - \alpha z$ is the flow benefit from the unrestricted component of the subsidy;
2. $\alpha(1-\phi)(k-z)$ is the firm's net-of-subsidy total cost of k units of R&D effort;
3. $[\lambda kp\Pi + (1-\lambda kp)V(p)]$ is the expected benefit from k units of R&D effort;
4. $\lambda kp(1-p)V'(p)$ is the expectation, with k units of R&D effort, of the option value of waiting that is foregone if the firm achieves a breakthrough.

Proposition 1 characterizes the firm's optimal R&D strategy.

Proposition 1 *Given a subsidy policy* $\mathbf{a} = (z, s, \phi)$, *let* $p_1(\mathbf{a})$ *be the solution to*[14]

$$\lambda p \left[\Pi - \frac{(s - \alpha z)}{r} \right] \frac{r}{r + \lambda z} - (1-\phi)\alpha = 0, \tag{7}$$

[13] The approach is to form a Taylor expansion of $V(p)$ (ignoring the higher order terms which disappear as dt goes to 0), which gives

$$V(p+dp) = V(p) + V'(p)\,dp = V(p) - \lambda kp(1-p)V'(p)\,dt.$$

Substituting this into (5), taking limits as $dt \to 0$, and simplifying yields (6).

[14] Because $\frac{\bar{s}}{r} < \Pi$, it follows that $\Pi - \frac{s-\alpha z}{r} > 0$ for all feasible s and z, and thus the solution to (7) is such that $p > 0$.

that is

$$p_1(\mathbf{a}) = \frac{\alpha\,(1-\phi)}{\lambda\left[\Pi - \left(\frac{s-\alpha z}{r}\right)\right]\left(\frac{r}{r+\lambda z}\right)}. \tag{8}$$

Then for any arbitrary belief $p \in [0,1]$*, the monopolist's optimal R&D strategy is*

$$k_1(p) = \begin{cases} 1 & \text{if } p > p_1(\mathbf{a}) \\ z & \text{if } p \leq p_1(\mathbf{a}) \end{cases},$$

and its value function is

$$V_1(p) = \begin{cases} \frac{s-(1-\phi)\alpha-\alpha\phi z}{r} + \frac{\lambda}{r+\lambda}\left(\Pi - \left[\frac{s-(1-\phi)\alpha-\alpha\phi z}{r}\right]\right)p + B_1(\mathbf{a})p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}} & \text{if } p > p_1(\mathbf{a}) \\ \frac{s-\alpha z}{r} + \frac{\lambda z}{r+\lambda z}\frac{r\Pi-(s-\alpha z)}{r}p & \text{if } p \leq p_1(\mathbf{a}) \end{cases},$$

where

$$B_1(\mathbf{a}) \triangleq \frac{\lambda\alpha\,(1-z)\,(1-\phi)}{r\,(r+\lambda)}\left(\frac{1-p_1(\mathbf{a})}{p_1(\mathbf{a})}\right)^{-\frac{r}{\lambda}}.$$

According to Proposition 1, the monopolist's optimal investment decision $k_1(p)$ is a "bang-bang" rule: k either equals the minimum required level z, or the maximum feasible level 1, depending on whether its belief p about the project's viability is greater or less than $p_1(\mathbf{a})$. We refer to $p_1(\mathbf{a})$ as the *abandonment threshold* because it is the point at which the firm abandons a "flat out" commitment to the R&D program.

A firm's realized (i.e., equilibrium path) behavior under the optimal policy depends on its prior belief about the project's viability, p_0. If the firm is sufficiently optimistic about the project's viability *ex ante* so that $p_1(\mathbf{a}) < p_0$, there exists a non-empty interval $(p_1(\mathbf{a}), p_0]$ of posterior beliefs p over which it exerts maximum R&D effort (i.e., $k = 1$), an interval we refer to as the *range of maximum investment.*[15] In this case, the firm begins by investing flat out. As time passes without a breakthrough, it becomes more pessimistic about the viability of the project, but as long as the posterior falls within the range of maximum investment, the firm continues to exert maximum R&D effort. Once its posterior falls to $p_1(\mathbf{a})$, the firm switches from "flat out" investment to the minimum level mandated by the government (i.e., $k = z$). If, by contrast, the firm is sufficiently pessimistic about the viability of the project *ex ante* so that $p_0 \leq p_1(\mathbf{a})$, the range of maximum investment is empty, and the firm begins by exerting the

[15] Clearly, this case cannot arise if $p_1(\mathbf{a}) \geq 1$.

minimum mandated level z and continues doing so as long as a breakthrough does not occur.

The abandonment threshold comes from a "marginal cost equals marginal benefit" condition (7). The marginal cost of an additional unit of R&D effort above the minimum threshold is $(1-\phi)\alpha$. The matching component of the subsidy ϕ thus reduces the firm's marginal cost. The firm's marginal benefit equals $\lambda p\left(\Pi - \frac{s-\alpha z}{r}\right)\frac{r}{r+\lambda z}$. The marginal benefit consists of two components: (a) the incremental increase in the likelihood of a breakthrough, λp, and (b) the net prize to the firm if a breakthrough occurs, $\left(\Pi - \frac{s-\alpha z}{r}\right)\frac{r}{r+\lambda z}$. The firm's net prize is the present value of profits from a breakthrough, Π, minus the present value of the fungible portion of the subsidy, $\frac{s-\alpha z}{r}$, which the firm foregoes if it achieves a breakthrough. The net prize is further "deflated" by the term $\frac{r}{r+\lambda z}$ which is less than 1 when $z > 0$; as z increases the extent of this deflation increases, and the firm's marginal benefit falls.[16]

The incentive properties of the three policy instruments follow immediately from (8). Suppose $\mathbf{a}$ and p_0 are such that $p_1(\mathbf{a}) < p_0$. Then

- Holding s and z fixed, an increase in the matching rate ϕ decreases the abandonment threshold, thus expanding the range of maximum investment;
- Holding ϕ and z fixed, an increase in the baseline subsidy s (which thus increases the unrestricted portion of the subsidy $s - \alpha z$) increases the abandonment threshold, thus contracting the range of maximum investment;
- Holding ϕ fixed and the unrestricted portion of the subsidy $s - \alpha z$ fixed, an increase in the mandated minimum z increases the abandonment threshold, thus contracting the range of maximum investment.

Suppose, by contrast, $\mathbf{a}$ and p_0 are such that $p_1(\mathbf{a}) \geq p_0$, so there is no range of maximum investment. Then, holding s and z fixed, a sufficiently large increase in the matching rate ϕ could give rise to a

[16] Formally, it can be shown that $\frac{r}{r+\lambda z}$ is the arc elasticity of the expected present value of dollar with respect to z. Specifically:

$$\frac{r}{r+\lambda z} = \frac{\frac{E(e^{-r\widetilde{T}}|k=1)-E(e^{-r\widetilde{T}}|k=z)}{E(e^{-r\widetilde{T}}|k=1)}}{\frac{1-z}{1}},$$

where $\widetilde{T}$ denotes the time to discovery; $E(e^{-r\widetilde{T}}|k=1) = \int_0^\infty e^{-rt}\lambda e^{-\lambda t}dt = \frac{\lambda}{\lambda+r}$ is the expected present value of \$1 when the firm invests flat out; and $E(e^{-r\widetilde{T}}|k=z) = \int_0^\infty e^{-rt}\lambda z e^{-\lambda z t}dt = \frac{\lambda z}{\lambda z+r}$ is the expected present value of \$1 when the firm invests at level $z \in [0,1]$. These expressions arise because, conditional on the project being viable, if the investment effort is a constant k, then discovery time is an exponetial random variable with parameter λk. Thus, $\frac{r}{r+\lambda z}$ represents (approximately) the percentage change in the expected value of \$1 worth of a prize per one pecent change in R&D effort above the minimum level. When $z = 0$, this elasticity equals 1, As the mandated minimum increases, this elasticity decreases.

non-empty range of maximum investment, but increases in either s or z (holding ϕ fixed) would not do so—the firm would continue to invest at the minimum mandated level.

Both the unrestricted component of the subsidy s and the minimum mandate z are a drag on R&D incentives. By investing more heavily and accelerating the expected time to a breakthrough, the firm brings to an end more quickly the flow $s - \alpha z$ of fungible benefits. Increases in unrestricted funding magnify this negative consequence. Furthermore, the minimum mandate z itself becomes a drag on R&D incentives for a different reason. When $k = z$, the firm, in effect, receives a fully-funded option from the government: the government is paying for the R&D investment z, but the firm receives the prize Π if a discovery is made. A firm faces a trade-off between accelerating the time to breakthrough at its own cost and retaining the free option from the government. A larger z increases the option value and thus reduces a firm's incentives to use its own resources for R&D.

We reinforce these insights by considering the special cases of a pure matching subsidy, an earmarked subsidy, and a pure unrestricted subsidy, which have been defined in Section 2, and comparing the outcomes in those cases to the case in which the firm does not receive a subsidy. Using (8), when the firm does not receive a subsidy, its abandonment threshold is

$$p_1^{NO} = \frac{\alpha}{\lambda \Pi}.$$

The abandonment thresholds p_1^M, p_1^E, and p_1^U for a pure matching subsidy, earmarked subsidy, and unrestricted subsidy are respectively

$$p_1^M = \frac{(1-\phi)\,\alpha}{\lambda \Pi}.$$

$$p_1^E = \frac{\alpha}{\lambda \Pi \left(\frac{r}{r+\lambda z}\right)}.$$

$$p_1^U = \frac{\alpha}{\lambda \left(\Pi - \frac{s}{r}\right)}.$$

From these expressions we see that

- $p_1^M \leq p_1^{NO}$, i.e., with the pure matching subsidy, the firm invests in the R&D project for at least as long as it would have in the absence of a subsidy and strictly longer if $\frac{(1-\phi)\alpha}{\lambda \Pi} < p_0$. If $p_0 \in \left(\frac{(1-\phi)\alpha}{\lambda \Pi}, \frac{\alpha}{\lambda \Pi}\right)$, a pure matching subsidy induces the firm to invest in a project that a non-subsidized firm would not have.

- $p_1^E \geq p_1^{NO}$, i.e., with an earmarked subsidy, the firm stops funding R&D from its own resources at least as soon as it would have without a subsidy and strictly sooner if $p_0 > \frac{\alpha}{\lambda\Pi}$. If $p_0 \in \left(\frac{\alpha}{\lambda\Pi}, \frac{\alpha}{\lambda\Pi\left(\frac{r}{r+\lambda z}\right)}\right)$ then net-of-subsidy R&D spending by a firm receiving an earmarked subsidy would be zero, while R&D spending by a non-subsidized firm would be positive. In other words, the subsidy would crowd-out the private R&D investment.

- $p_1^U \geq p_1^{NO}$, i.e., with a pure unrestricted subsidy, the firm stops investing in the R&D project at least as soon as it would have in the absence of a subsidy and strictly sooner if $p_0 > \frac{\alpha}{\lambda\Pi}$. If $p_0 \in \left(\frac{\alpha}{\lambda\Pi}, \frac{\alpha}{\lambda\left[\Pi - \frac{s}{r}\right]}\right)$ a pure unrestricted subsidy induces the firm to shut down investment in a project that a non-subsidized firm would have continued to fund.

It is important to note that even though increases in the matching rate decrease the abandonment threshold and thus expand the range of maximum investment, the impact of the matching subsidy on cumulative R&D investment need not be especially "cost effective." We can see this by considering a setting in which the prior p_0 is extremely close to 1. If $p_1^{NO} < 1$, an unsubsidized firm would initially invest in R&D, and since the posterior p would initially evolve very slowly (because the firm is virtually certain that the project is viable), a long period of time would have to pass with no breakthrough before the firm abandoned its efforts. A matching subsidy would indeed expand the time the firm persisted with the project, but for a significant period of time the government would be reimbursing some of the firm's R&D expenses even though the firm would have invested in R&D even had it not received that reimbursement over this time frame. As we discuss below, a consideration of this sort becomes potentially relevant if the financing of subsidies entails a positive shadow cost of public funds.

3.2 R&D by Multiple Firms ($N > 1$)

We now consider the case in which N firms compete to achieve the R&D breakthrough. With N firms, strategic behavior introduces the possibility of free-riding, which in turn influences how subsidy policy shapes incentives.

The value function of firm i, $i = 1, .., N$, is given by the recursion

$$V^i(p) = \max_{k^i \in [z,1]} \left[\begin{array}{c} \left(s - \alpha\phi z - \alpha(1-\phi) k^i\right) dt + \lambda p k^i dt \Pi + \lambda p K^{-i} dt \theta \Pi \\ + \left(1 - \lambda p \left(k^i + K^{-i}\right) dt\right) e^{-rdt} V^i(p + dp) \end{array} \right], \tag{9}$$

where $K^{-i} = \sum_{j \neq i} k^j$ is the sum of the R&D investments by firm i's rivals. We can rewrite the value function in (9) as a differential equation:

$$(1+r)V^i(p) = s - \alpha z + \max_{k^i \in [z,1]} \left\{ \begin{array}{c} \left[\lambda k^i p \Pi + \lambda p K^{-i} \theta \Pi + (1 - \lambda\left(k^i + K^{-i}\right) p) V^i(p)\right] \\ -\alpha(1-\phi)(k-z) - \lambda\left(k^i + K^{-i}\right) p(1-p) V^{i\prime}(p) \end{array} \right\}. \tag{10}$$

As in the case of $N = 1$, a firm's value has four components. Two of these components are identical to the monopoly case; the other two differ due to the presence of competitors who can also achieve the breakthrough:

1. $s - \alpha z$ is the flow benefit from the unrestricted component of its subsidy;

2. $\alpha(1-\phi)(k-z)$ is the firm's net-of-subsidy total cost of k^i units of R&D effort;

3. $\left[\lambda k^i p \Pi + \lambda p K^{-i} \theta \Pi + (1 - \lambda\left(k^i + K^{-i}\right) p) V^i(p)\right]$ is the expected benefit to a firm from its own k^i units of R&D effort and its rivals' collective R&D effort K^{-i}.

4. $\lambda\left(k^i + K^{-i}\right) p(1-p) V^{i\prime}(p)$ is the expectation, with k^i+ K^{-i} units of R&D effort in total, of the option value of waiting that is foregone if some firm achieves a breakthrough.

Throughout, we focus on the symmetric equilibrium, i.e., $k^1 = \ldots = k^N = k$, which we characterize as follows:

Proposition 2 *Let*

$$p_N(\mathbf{a}) = \frac{\alpha(1-\phi)}{\lambda\left(\left[\Pi - \frac{(s-\alpha z)}{r}\right]\left(\frac{r}{r+\lambda N z}\right) + (N-1)(1-\theta)\Pi\left(\frac{\lambda z}{r+\lambda N z}\right)\right)}; \tag{11}$$

and define

$$\overline{\theta}(z,s) \triangleq 1 - \frac{\Pi - \left(\frac{s-\alpha z}{r}\right)}{\Pi} \frac{r}{r+\lambda z} \in [0,1]. \tag{12}$$

(i) If the spillover θ exceeds the critical level $\overline{\theta}(z,s)$, there exists a unique symmetric Markov perfect equilibrium with the following strategy:[17]

$$k_N(p) = \begin{cases} 1 & \textit{if } p \geq q_N(\mathbf{a}) \\ \frac{rV_N^M(p)-s+\alpha\phi z}{(N-1)(\alpha(1-\phi)-\lambda p(1-\theta)\Pi)} & \textit{if } p_N(\mathbf{a}) > p > q_N(\mathbf{a}) \\ z & \textit{if } p \leq p_N(\mathbf{a}) \end{cases},$$

and an individual firm's value function is

$$V_N(p) = \begin{cases} \frac{s-\alpha(1-\phi)-\alpha\phi z}{r} + \frac{\lambda N\left(\frac{r(1-\theta)\Pi}{N}+r\theta\Pi-s+\alpha(1-\phi)+\alpha\phi z\right)}{r(r+\lambda N)}p + B_H(\mathbf{a})p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda N}{\lambda N}} & \textit{if } p \geq q_N(\mathbf{a}) & (V_{N1}) \\ \frac{\lambda\Pi-(1-\phi)\alpha}{\lambda} - B_M(\mathbf{a})(1-p) + \frac{(1-\phi)\alpha(1-p)}{\lambda}\ln\frac{1-p}{p} & \textit{if } p_N(\mathbf{a}) < p < q_N(\mathbf{a}) & (V_{N2}) \\ \frac{s-\alpha z}{r} + \frac{\lambda Nz\left[\frac{r(1-\theta)\Pi}{N}+r\theta\Pi-s+\alpha z\right]}{r(r+\lambda Nz)}p & \textit{if } p \leq p_N(\mathbf{a}) & (V_{N3}) \end{cases}$$

where $B_M(\mathbf{a})$ equates (V_{N2}) and (V_{N3}) at $p = p_N(\mathbf{a})$; $q_N(\mathbf{a})$ satisfies

$$\frac{r\left[\frac{\lambda\Pi-(1-\phi)\alpha}{\lambda} - B_M(\mathbf{a})(1-q_N) + \frac{(1-\phi)\alpha(1-q_N)}{\lambda}\ln\frac{1-q_N}{q_N}\right] - s + \alpha\phi z}{(N-1)(\alpha(1-\phi)-\lambda q_N(1-\theta)\Pi)} = 1, \tag{13}$$

and $B_H(\mathbf{a})$ equates (V_{N1}) and (V_{N2}) at $p = q_N(\mathbf{a})$.

(ii) If the spillover θ is below the critical level $\overline{\theta}(z,s)$, there exists a unique symmetric Markov perfect equilibrium with the following strategy:

$$k_N(p) = \begin{cases} 1 & \textit{if } p > p_N(\mathbf{a}) \\ z & \textit{if } p \leq p_N(\mathbf{a}) \end{cases},$$

with the value function

$$V_N(p) = \begin{cases} \frac{s-\alpha(1-\phi)}{r} + \frac{\lambda N\left(\frac{r(1-\theta)\Pi}{N}+r\theta\Pi-s+\alpha(1-\phi)\right)}{r(r+\lambda N)}p + B_H^*(\mathbf{a})p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda N}{\lambda N}} & \textit{if } p > p_N(\mathbf{a}) & (V_{N4}) \\ \frac{s-\alpha z(1-\phi)}{r} + \frac{\lambda Nz\left[\frac{r(1-\theta)\Pi}{N}+r\theta\Pi-s+\alpha z(1-\phi)\right]}{r(r+\lambda Nz)}p & \textit{if } p \leq p_N(\mathbf{a}). & (V_{N5}) \end{cases},$$

where $B_H^*(\mathbf{a})$ equates (V_{N4}) and (V_{N5}) at $p = p_N$.

Panel (a) of Figure 1 illustrates the equilibrium investment policy when the spillover parameter θ

[17] Recall that the subscript N denotes the N-firm case.

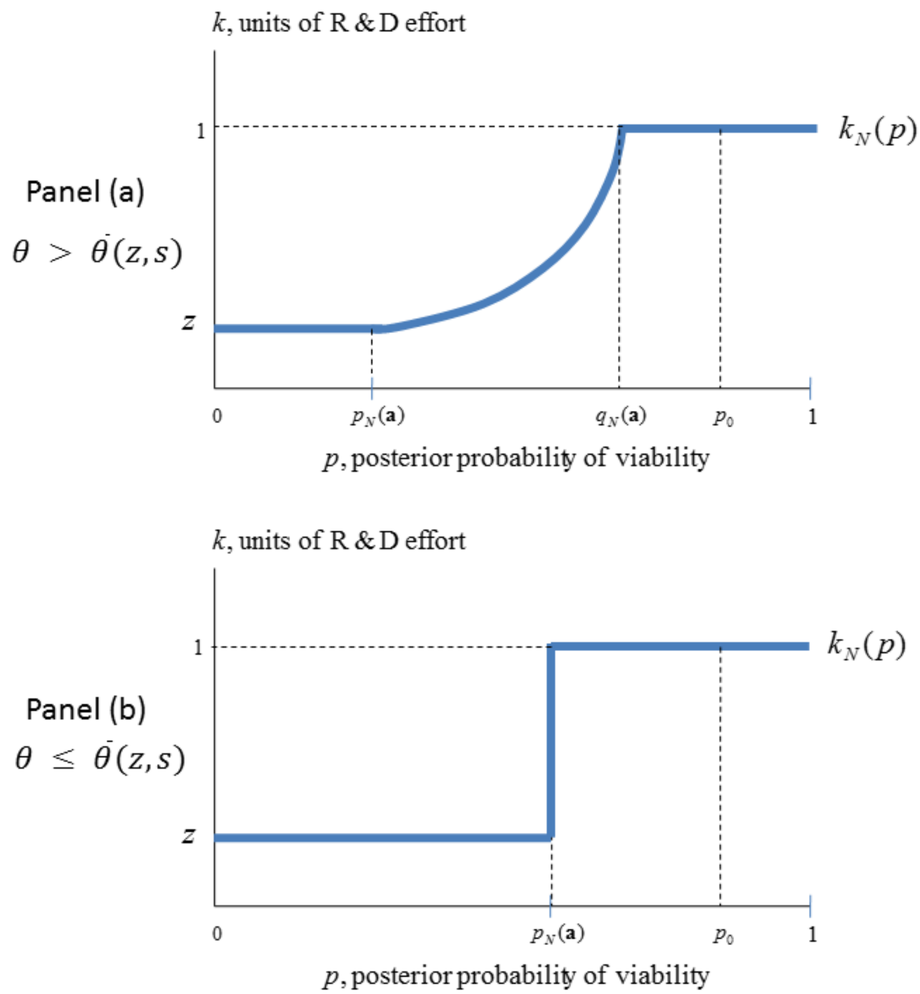

Figure 1: Equilibrium investment strategy with N firms.

exceeds the critical level $\overline{\theta}(z, s)$. In contrast to monopoly, in which the equilibrium investment policy is "bang-bang," there is a range of beliefs $(p_N(\mathbf{a}), q_N(\mathbf{a}))$ over which $k_N(p) \in (z, 1)$. If the posterior belief falls to the slowdown threshold $q_N(\mathbf{a})$, firms start to taper off their research efforts by reducing k below 1. If the posterior reaches the abandonment threshold $p_N(\mathbf{a})$, a firm chooses the minimum required R&D effort, $k_N(p) = z$.

The equilibrium involves $k_N(p) \in (z, 1)$ because of a free-rider problem. The free-rider problem arises because the firm can achieve a positive payoff $\theta\Pi$ from spillover even if it loses the R&D competition, a phenomenon that does not arise under monopoly. In particular, when $\theta > \overline{\theta}(z, s)$ and $p \in (p_N(\mathbf{a}), q_N(\mathbf{a}))$, given that all other firms invest "flat out," it will be optimal for a firm to reduce its R&D investment below the maximum level. On the other hand, though, given that all other firms invest at the minimum level, it will be optimal for a firm to invest "flat out" in R&D. The "concession" to the free-rider problem that is made in equilibrium is that for $p \in (p_N(\mathbf{a}), q_N(\mathbf{a}))$, all firms reduce k to a positive number less than 1. The equilibrium value of $k_N(p)$ is such that when a firm's $N-1$ competitors invest $k_N(p)$, it is indifferent in investing among all $k \in (z, 1)$ and thus chooses $k_N(p)$ in a symmetric equilibrium.

By contrast, when the spillover parameter is less than $\overline{\theta}(z, s)$, the free-rider problem does not arise, and as shown in panel (b) of Figure 1, the equilibrium investment policy is "bang-bang," as in the case of a monopoly. However, the abandonment threshold $p_N(\mathbf{a})$ may not correspond to the monopoly threshold $p_1(\mathbf{a})$. In fact, for any subsidy policy such that $z > 0$, then using (8) and (11), we have[18]

$$\frac{p_1(\mathbf{a})}{p_N(\mathbf{a})} = \frac{\lambda\left(\left[\Pi - \frac{(s-\alpha z)}{r}\right]\frac{r}{r+\lambda N z} + (N-1)(1-\theta)\,\Pi\frac{\lambda z}{r+\lambda N z}\right)}{\lambda\left[\Pi - \left(\frac{s-\alpha z}{r}\right)\right]\frac{r}{r+\lambda z}} > 1, \tag{14}$$

i.e., the range of maximum investment is greater with N firms than with a monopolist. The intuition is this. The fully funded option that z provides when a firm ceases R&D investment above the minimum mandate becomes less valuable when it competes with $N-1$ other firms, each of who could potentially get the prize from the breakthrough. As a result, the minimum mandate z is less of a drag on R&D incentives with N firms than with a single firm.

Because $\overline{\theta}$ depends on z and s, whether or not free riding arises in equilibrium depends on the subsidy policy. We note that if there is neither unrestricted funding nor minimum investment mandate — i.e., $z = s = 0$ — then $\overline{\theta}(z, s) = 0$, and the free-rider problem always arises. This tells us that a necessary condition for avoiding the free-rider problem is to establish a mandated minimum level of R&D ($z > 0$) or provide positive baseline funding ($s > 0$), or both. To understand why, recall from the discussion of monopoly that a subsidy with an unrestricted component $s - \alpha z$ creates an implicit loss for the firm when it achieves a breakthrough. By the same token, an unrestricted component creates an implicit loss when *another firm* achieves a breakthrough. Thus, the unrestricted component of the subsidy offsets part of the gain the firm receives when another firm makes the discovery, thereby reducing the temptation to free ride. A subsidy policy with a minimum mandate also creates an implicit loss for the firm when another firm achieves the breakthrough, but for a different reason. To see why, note that when $k = z$, the firm is receiving an option (the possibility that it, or another firm, achieves a breakthrough) that is fully paid for by the government. When another firm achieves a breakthrough, this government-funded

[18] Recall that $\Pi > \frac{(s-\alpha z)}{r}$ given our parameter assumptions, so the condition in (14) is meaningful.

option goes away, creating an implicit loss that can offset some of the gains from free riding. Thus, in contrast to the monopoly case, in which increases in s and z had unambiguously adverse effects on the provision of R&D, under R&D competition s and/or z may have potentially beneficial incentive effects by mitigating the extent of free riding behavior by firms.

Each of the policy choices affects investment incentives through the entire equilibrium strategy $k_N(p)$. These effects cannot be determined analytically, but the expression for the abandonment threshold in (11) has these implications:

- Holding s and z fixed, an increase in the matching rate ϕ decreases the N-firm equilibrium abandonment threshold, thus expanding the range over which firm invests in excess of the mandated minimum;

- Holding ϕ and z fixed, an increase in the baseline subsidy s (which thus increases the unrestricted component of the subsidy $s - \alpha z$) increases the N-firm equilibrium abandonment threshold, thus contracting the range over which the firm invests in excess of the mandated minimum.

This discussion hints at an interesting tension involving the baseline subsidy s. On the one hand, it can crowd out private R&D investment, by contracting the range over which the firm invests in excess of the mandated minimum. On the other hand, it can counteract the free-rider problem. In the next section, we will see that in designing a subsidy scheme when the shadow cost of public funds is 0, the government can exploit this tension.

Summarizing the impact of subsidies in the N-firm case, if s and z are sufficiently large, the free-rider problem will not arise in equilibrium. However, changes in ϕ have no impact on free riding. As in the monopoly case, increases in ϕ decrease the abandonment threshold $p_N(\mathbf{a})$ (thus expanding the range of private funding of R&D), while increases in s increase the abandonment threshold. Unlike the monopoly case, changes in z have an ambiguous impact on the abandonment threshold. The impact of z, s, and ϕ on the slowdown threshold $q_N(\mathbf{a})$ and the equilibrium investment policy $k_N(p)$ more generally cannot be determined analytically.

The lesson of this section is that the way in which R&D is subsidized matters. Certain types of subsidies (i.e., earmarked subsidies in the case of $N = 1$) may crowd out private investment, while other types of subsidies (i.e., a pure matching subsidy in which a firm undertaking R&D is reimbursed a fraction of its R&D expenses) may stimulate private investment. This suggests that empirical studies of the impact of R&D subsidies on private R&D investment need to be cognizant of the subsidy mechanism.

4 Optimal Subsidy Policy Under "If" and "When" Uncertainty For an Informationally Constrained Policy Maker

In the previous section, we have shown that a firm's incentive to invest in R&D depends on how R&D subsidies are structured. We now turn to the question of the best subsidy policy. Recall our assumption that the government neither knows the prior belief p_0 nor has probabilistic beliefs describing its likelihood, which prevents the government from offering a contract that ties the parameters of the subsidy with the posterior belief that determines a firm's R&D strategy. Again, we begin with monopoly R&D and then turn to R&D competition.

4.1 Monopoly R&D: $N = 1$

As a benchmark, we begin by noting that the first-best R&D investment policy when $N = 1$ is[19]

$$k^*(p) = \begin{cases} 1 & \text{if } p > p^* \\ 0 & \text{if } p \leq p^* \end{cases}, \tag{15}$$

where

$$p^* = \frac{\alpha}{\lambda\,(CS + \Pi)} < 1. \tag{16}$$

The first-best abandonment threshold p^* is the reciprocal of the social benefit-cost ratio $\frac{\lambda(CS+\Pi)}{\alpha}$. Note that $p^* < p_1^{NO}$, so under the first-best policy, investment in the R&D project continues at least as long as it would have in an unsubsidized firm, and strictly longer if $p^* < p_0$, i.e., whenever there is any investment in the first-best solution. If $p_0 \in \left(p^*, \frac{\alpha}{\lambda\Pi}\right)$, the first-best policy entails investment for some length of time, while the unsubsidized firm opts for no investment at all. Thus, an unsubsidized firm underinvests relative to the first-best level.

If the government knew the firm's prior p_0 (and thus the subsequent posterior beliefs p), it could direct the firm to follow the policy in (15) and (16) and achieve the first-best welfare level. Our assump-

[19] Derivation of this policy is a straightforward extension of the proof of Proposition 1 in which we set $s = z = \phi = 0$, and replace Π with $CS + \Pi$.

tion that the government does not know p_0 rules this out. Instead, we assume that the government must rely on subsidies to provide incentives to the firm. Throughout the analysis, we assume that the subsidy is funded by revenues from broad-based taxes that do not materially affect the firm's incentive to invest in R&D. However, we do allow for the possibility of a shadow cost of public funds $\gamma \geq 0$. That is, a subsidy S to the firm entails a transfer of S from taxpayers plus a social cost γS, where $\gamma \geq 0$.

In addition to having a social cost, a subsidy policy could also be distortionary in the sense that it could induce underinvestment or overinvestment in R&D relative to the first-best level. Consider, for example, a policy $\mathbf{a}$ such that $p_1(\mathbf{a}) < p^*$ (which would be due to an especially generous matching rate). If $p_0 \in [p_1(\mathbf{a}), p^*]$, the first-best policy calls for no investment at all, while the subsidy policy induces the firm to invest "flat-out" for at least some period of time, and if $z > 0$, the firm would continue to invest once its posterior fell to $p_1(\mathbf{a})$. In this case, we have (possibly rather significant) overinvestment. On the other hand, for a subsidy policy such that $p^* < p_1(\mathbf{a})$, if $p_0 \in [p^*, p_1(\mathbf{a})]$, the firm would spend none of its own funds investing (though if $z > 0$, it would invest at the minimum mandated level), while the first-best solution would have given rise to non-empty maximum investment range. In this case, then, we would have underinvestment.

Before turning to consideration of optimal subsidy policy under incomplete information, we note that subsidies with a shadow cost of public funds can potentially create subtle trade-offs between the various policy instruments, trade-offs that depend on the firm's private information. To illustrate, suppose that the firm is virtually certain that the project is viable so that p_0 is extremely close to 1. As discussed above, if $p_1^{NO} < 1$, the firm would be willing to invest for quite a while even if it did not receive a subsidy. A pure matching subsidy $(0, 0, \phi)$ would extend the range of maximum investment and thus could have potential welfare benefits for the government in the eventuality that a long time passes without a breakthrough. However, it might not have much impact in the near, or even intermediate, term (since $p(t)$ would remain close to p_0 for quite a while), and if $\gamma > 0$, this could prove to be a costly way to motivate the firm because it transfers socially valuable funds to the firm to do what (in the near term, at least) it would have done anyway. By contrast, an earmarked subsidy $(z, \alpha z, 0)$ would have the drawback that it would reduce the range of maximum investment. However, for z sufficiently small, $p_1^E < 1$ and since (by assumption) $p_0 \approx 1$, the firm would invest flat out at least for a while (and it would continue to invest a modest amount z thereafter). With a positive shadow cost of public funds, it is *conceivable* that the government might prefer the potentially smaller subsidy payments that it would make under the earmarked subsidy to the bigger payments it might make under the pure matching subsidy, even though the former policy has the effect of suppressing R&D incentives to some extent.

4.1.1 Optimal R&D Policy Under Incomplete Information: $N = 1$

What subsidy policy would the government choose in the face of incomplete information about the prior p_0? To develop an answer to this question, we begin by deriving the expression for expected social welfare induced by the firm's optimal investment rule $k_1(p)$ derived in Proposition 1. To do so, we note that if the firm invests $k_1(p)$ units in the R&D project for the time interval $[t, t+dt)$ when the belief about the project's viability is p, the social welfare schedule $W_1(p)$ is given recursively as follows:

$$\begin{aligned} W_1(p) &= -\alpha k_1(p)dt - \gamma\left[s + \phi\alpha(k_1(p) - z)\right]dt \\ &\quad + \lambda k_1(p)pdt\left[CS + \Pi\right] + (1 - \lambda k_1(p)pdt)e^{-rdt}W_1(p + dp). \end{aligned}$$

This recursion can be transformed into the following differential equation:

$$0 = -\alpha k_1(p) - \gamma\left[s + \phi\alpha(k_1(p) - z)\right] + \lambda k_1(p)p\left(CS + \Pi\right) - \left(r + \lambda k_1(p)p\right)W(p) - \lambda k_1(p)p(1-p)W'(p). \tag{17}$$

In the Appendix, we show that the solution to this differential equation is

$$W_1(p) = \begin{cases} -\left(\frac{\alpha + \gamma s + \gamma\phi\alpha(1-z)}{r}\right) + \frac{\lambda}{r+\lambda}\left[CS + \Pi + \left(\frac{\alpha + \gamma s + \gamma\phi\alpha(1-z)}{r}\right)\right]p + B_W p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}} & \text{if } p > p_1(\mathbf{a}) \\ -\left(\frac{\alpha z + \gamma s}{r}\right) + \frac{\lambda z}{r+\lambda z}\left[CS + \Pi + \left(\frac{\alpha z + \gamma s}{r}\right)\right]p & \text{if } p \leq p_1(\mathbf{a}) \end{cases}, \tag{18}$$

where $p_1(\mathbf{a})$ is the abandonment threshold in (8), and B_W makes the welfare schedule continuous at $p = p_1(\mathbf{a})$.[20]

The function $W_1(p)$ is expected social welfare for any arbitrary posterior belief p when the firm's investment strategy is a best response to subsidy policy $\mathbf{a}$. It follows that evaluating $W_1(p)$ at $p = p_0$ gives us the government's expected social welfare if it offered policy $\mathbf{a}$ and *it knew the firm's prior belief* p_0. We call this *conditional expected social welfare* (i.e., conditioned on p_0), and we denote it by

[20] Continuity of the welfare schedule follows because at $p = p_1(\mathbf{a})$, the firm is indifferent between $k = z$ and $k = 1$.

$W_1(p_0|\mathbf{a})$. Specifically,

$$W_1(p_0|\mathbf{a}) = \begin{cases} \Psi(B_W(p_1(\mathbf{a}),\mathbf{a}),\mathbf{a}) & p_0 \in [p_1(\mathbf{a}),1] \\ -\left(\frac{\alpha z+\gamma s}{r}\right) + \frac{\lambda z}{r+\lambda z}\left[CS + \Pi + \left(\frac{\alpha z+\gamma s}{r}\right)\right] p_0 & p_0 \in [0,p_1(\mathbf{a})] \end{cases}, \tag{19}$$

where

$$\Psi(B_W,\mathbf{a}) \triangleq \left\{ \begin{array}{c} -\left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right) \\ +\frac{\lambda}{r+\lambda}\left[CS+\Pi+\left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)\right] p_0 \\ +B_W p_0 \left(\frac{1-p_0}{p_0}\right)^{\frac{r+\lambda}{\lambda}} \end{array} \right\},$$

$$B_W(p_1,\mathbf{a}) \triangleq \frac{\left\{ \begin{array}{c} \frac{\alpha(1-z)}{r} + \frac{\gamma\phi\alpha(1-z)}{r} - \\ \frac{\lambda}{r+\lambda}\left[CS+\Pi+\left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)\right] p_1 \\ +\frac{\lambda z}{r+\lambda z}\left[CS+\Pi+\left(\frac{\alpha z+\gamma s}{r}\right)\right] p_1 \end{array} \right\}}{p_1\left(\frac{1-p_1}{p_1}\right)^{\frac{r+\lambda}{\lambda}}},$$

and

$$p_1 = p_1(\mathbf{a}).$$

We emphasize that the government cannot actually condition on p_0 (since it does not know what it is), nor can it "pin down" an expectation of $W_1(p_0|\mathbf{a})$ with respect to a prior distribution over p_0 since we have assumed it does not have a unique prior. That said, it is extremely useful to consider the problem of choosing a subsidy policy to maximize conditional expected welfare:

$$\max_{\mathbf{a}\in A} W_1(p_0|\mathbf{a}), \tag{20}$$

Let $\mathbf{a}^{**}(p_0)$ denote the solution to this problem. *If* it was the case that the solution to this problem was independent of p_0—i.e., $\mathbf{a}^{**}(p_0) = \mathbf{a}^{**}$—then the government could implement $\mathbf{a}^{**}$ without requiring knowledge of the firm's private information, and it would be assured that it would attain the highest welfare it could possibly achieve whatever the firm's actual type p_0 might be. Theoretically, this situation would be a belief-free incomplete information game (see Bergemann and Morris, 2007), where each uninformed player (e.g., the government here) takes an agnostic view about opponents' unobserved types by refraining from assigning potentially restrictive probability measures on them (as conventional

Bayesian models do). In this case, the policy in question qualifies as an *ex post* equilibrium. Informally, an *ex post* equilibrium is such that no uninformed player will find it profitable to unilaterally deviate from her strategy even if her incomplete information (about opponents' types) is replaced by complete information. It is obvious to see that here, even if the government knows the true value of p_0, *ceteris paribus* $\mathbf{a}^{**}(p_0)$ would remain an optimal policy. The *ex post* equilibrium status attached to this policy gives it a robust justification in terms of information requirements.

We now show that when $\gamma = 0$—a subsidy is a pure transfer between taxpayers and the firm—an *ex post* equilibrium exists; it takes a strikingly simple form, and it attains first-best welfare.

Proposition 3 *When a single firm engages in R&D and if a subsidy is a pure transfer between consumers and firms, i.e.,* $\gamma = 0$*, then the following subsidy policy solves the maximization problem in (20) and thus constitutes an ex post equilibrium:*

$$\mathbf{a}^{**}(p_0) = \mathbf{a}^{**} = (0, 0, 1 - \rho) \textit{ for all } p_0 \in [0, 1],$$

where $\rho = \frac{\Pi}{\Pi + CS}$*. This policy induces the firm to choose an investment policy* $k_1(p) = k^*(p)$ *and thus achieves the first-best level of ex ante welfare for any prior belief* p_0*.*

Proposition 3 implies that when subsidies have no shadow cost, there is a simple way to achieve the first-best outcome: use a pure matching subsidy whose matching rate is 1 minus the appropriability ratio. No mandated minimum level of R&D is necessary, and it does not require any knowledge of the firm's private information for its implementation by the government.

Does this result generalize to the case in which the shadow cost of public funds γ is positive? The answer is no: the first-best policy cannot be implemented through subsidies, and indeed, no *ex post* equilibrium exits for $\gamma > 0$. We formalize this in the following proposition.

Proposition 4 *If* $p_1^{NO} < 1$*, then for* $\gamma > 0$ *there exists no ex post equilibrium, i.e., the solution to the optimization in (20) depends on* p_0*.*[21]

The intuition for this result is as follows. When the shadow cost of public funds is positive, the government faces a trade-off between using a subsidy to induce more R&D and incurring higher social

[21] Numerical examples can show that the restriction $p_1^{NO} < 1$ (which is useful for simple analytical proof) is unnecessary for the non-existence result.

costs due to the subsidy, and this trade off delicately depends on the value of p_0. To see why, suppose that the government *knew* that the firm was completely certain of the project's viability, i.e., $p_0 = 1$. In this case, the subsidy policy that solves the optimization in (20) is no subsidy at all. This is because (if $p_1^{NO} < 1$) the government would be certain that an unsubsidized firm would invest in R&D, no matter how much time passed without a breakthrough. (Recall that when $p_0 = 1$, there is no "if" uncertainty, and thus $p(t) = 1$ for all t.) Thus, a socially costly subsidy would have no impact on the firm's investment incentives. The problem, however, is the no-subsidy policy $\mathbf{a} = (0, 0, 0)$, while welfare maximizing if the government was certain that $p_0 = 1$, could be a poor policy if the government believed that the firm's prior was something else.

The non-existence problem limits the general applicability of the solution concept of *ex post* equilibrium. For this reason, we now open a discussion of potentially suitable solution concepts for our policy problem at hand.

4.1.2 What Policy Should the Government Choose When $\gamma > 0$?

What criteria should the government use to determine a preferred subsidy policy when $\gamma > 0$? We proceed in two steps. First, we show that there exists a simple policy with the appealing feature that it neither generates underinvestment nor overinvestment. Second—and more formally—we consider two candidate decision criteria for the government which are suitable for belief-free incomplete information games. The first criteria is (incomplete information) rationalizability; the second is the max-min criterion. In our discussion we define these two criteria, explore their relationship and what they imply about policy, and finally we indicate our preferred criterion.

Solving the Problem of Underinvestment without Inducing Overinvestment We saw earlier that an unsubsidized firm has a tendency to underinvest (relative to the first best), but that by subsidizing the firm, the government could conceivably induce overinvestment. With the government lacking the ability to make fine-tuned trade-offs between policy instruments because it has neither complete nor probabilistic information about p_0, a plausible criterion for a "good" R&D subsidy policy would be one that solves the problem of underinvestment, while not inducing overinvestment. In this section, we show that there is such a policy. Indeed, that policy is a natural extension of the policy characterized in Proposition 3, and that it smoothly approaches that policy in the limit as $\gamma \to 0$.

Consider a matching rate constructed by generalizing the definition of ϕ^{**} to

$$\phi_{\gamma}^{**} \triangleq \frac{\frac{CS}{1+\gamma}}{\Pi + \frac{CS}{1+\gamma}}. \tag{21}$$

Clearly, $\phi_{\gamma}^{**} = 1 - \rho$ for $\gamma = 0$. Further, let

$$p_1\left(0, 0, \phi_{\gamma}^{**}\right) = p_{\gamma}^{**} \triangleq \frac{\alpha}{\lambda\left(\Pi + \frac{CS}{1+\gamma}\right)} \tag{22}$$

be the abandonment threshold under the subsidy policy $\left(0, 0, \phi_{\gamma}^{**}\right)$. We now note two key properties of the policy $\left(0, 0, \phi_{\gamma}^{**}\right)$:

$$p\lambda\left(\Pi + CS\right) - \alpha - \gamma\phi_{\gamma}^{**}\alpha \geq 0 \text{ iff } p \geq p_{\gamma}^{**}, \tag{23}$$

and

$$p\lambda\Pi - \left(1 - \phi_{\gamma}^{**}\right)\alpha \geq 0 \text{ iff } p \geq p_{\gamma}^{**}. \tag{24}$$

Note that the left hand sides of the first equalities in (23) and (24) are the flow payoffs for the government and the firm respectively. Therefore, under subsidy policy $\left(0, 0, \phi_{\gamma}^{**}\right)$ both the firm and the government have zero flow payoffs for all $k \in [0, 1]$ for $p = p_{\gamma}^{**}$, and their flow payoffs strictly increase in p. Note that the two zero flow payoff conditions imply that the "marginal cost equal to marginal benefit condition" is satisfied at the same p_{γ}^{**} for both the government and the firm. In addition, both the firm and the government have zero continuation values for $p = p_{\gamma}^{**}$. Thus, from the government's perspective, given that it has committed to pay a matching rate ϕ_{γ}^{**}, p_{γ}^{**} is the optimal abandonment threshold. And from the firm's perspective, given that it receives a pure matching subsidy ϕ_{γ}^{**}, p_{γ}^{**} is the privately optimal abandonment threshold. Thus, if the government offers the policy $\left(0, 0, \phi_{\gamma}^{**}\right)$, it can be assured that whatever the firm's private information it will invest in the way most preferred by the planner.

We note that the policy $\left(0, 0, \phi_{\gamma}^{**}\right)$ has the intuitive feature that the matching rate is smaller than ϕ^{**}, and this rate decreases as the shadow cost of public funds increases. As the shadow cost increases without bound, the matching rate goes to zero, i.e., the policy involves no subsidization whatsoever.

Candidate Decision Criteria for a Belief-Free Incomplete Information Games We now turn to more formal decision criteria for belief-free incomplete information games.[22] We first consider rationalizability. (See Pearce, 1984; Battiqalli and Sciniscalchi, 2003.)

Given any subsidy policy $\mathbf{a} \in A$, the firm has a unique best response (as given by the bang-bang rule, described in Proposition 1), which is rationalizable. This fact simplifies the definition of the rationalizability (solution concept) for our game, so we only need to define the government's rationalizable strategy.

Definition 1 *Let Σ_A denote the Borel sigma algebra over A, and $\Delta(\Sigma_A)$ the set of all probability measures (i.e., mixed strategies) over Σ_A. Let Σ be the Borel sigma algebra over $[0,1]$ and $\Delta(\Sigma)$ the set of all probability measures (i.e., probabilistic beliefs) over Σ. A mixed strategy $\sigma \in \Delta(\Sigma_A)$ is rationalizable if $\exists \mu \in \Delta(\Sigma)$ such that*

$$\sigma = \arg \max_{\sigma' \in \Delta(\Sigma_A)} \int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma'(\mathbf{a})\, d\mu(p).$$

Since the value function W is complicated, this definition of rationalizability is not particularly operational. Fortunately there exists a more operational equivalent definition expressed in terms of strict dominance. The following proposition states the equivalence result, which is an extension of the finite game result of Pearce (1984).

Proposition 5 *The (mixed strategy) policy $\sigma \in \Delta(\Sigma_A)$ is rationalizable if and only if σ is not strictly dominated in $\Delta(\Sigma_A)$, i.e., $\nexists \sigma'' \in \Delta(\Sigma_A)$ such that*

$$\int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma''(\mathbf{a})\, d\mu(p) > \int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma(\mathbf{a})\, d\mu(p)$$

for all $\mu \in \Delta(\Sigma)$.

In words, a rationalizable policy is a mixture over possible subsidy policies with the feature that there is no other mixture over subsidy policies that gives the government a strictly higher expected welfare for all conceivable beliefs $\mu(\cdot)$ over p_0. Rationalizable policies thus have the appealing feature that they

[22] Besanko, Tong, and Wu (2016) provide a discussion of belief-free incomplete information games that goes beyond that presented here, including an exploration of belief-free games in relation to other lines of literature on dyamic games of incomplete information.

exclude policies that are obviously "bad" in the sense of being strictly inferior for the government's conceivable beliefs.

To put this into further perspective, it is useful to turn to the max-min criterion. This criterion has been featured prominently in the literature on decision under ambiguity (for example, see Gilboa and Schmeidler, 1989). Ambiguity can, in general, be modeled as a set of multiple probabilistic beliefs, allowing the flexibility to model belief-free incomplete information games, to which the concept of max-min solution can be applied. In a policy making context, max-min would correspond to a setting in which, although the government does not know *what to believe* about the firm's private information, it wishes to follow a policy whose "worst case scenario" outcome is not inferior to any alternative policy. Max-min is a potentially appealing decision criterion when policy makers (as often seems plausible) are especially attuned to the need to avoid potentially severe policy mistakes.

Definition 2 *The (mixed strategy) policy* $\sigma \in \Delta(\Sigma_A)$ *is a max-min solution if*

$$\sigma = \arg \max_{\sigma' \in \Delta(\Sigma_A)} \min_{\mu \in \Delta(\Sigma)} \int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma'(a)\, d\mu(p).$$

How does the max-min criterion relate to rationalizability? The following proposition establishes that being a max-min solution is a sufficient condition for rationalizability, i.e., the max-min solution concept in our setting is no more encompassing (and conceivably less so) than rationalizability.

Proposition 6 *All max-min solutions are rationalizable.*

Does the converse hold true, which would then imply that the set of max-min solutions and rationalizable solutions are one in the same? To answer this question, we first derive the set of all max-min solutions, which is also a subset of all non-dominated policies. Then we check whether we can find a counter example, i.e., a policy that does not belong to this subset but is not strictly dominated by any policy in this subset. If such an example can be found then it can be inferred that not all rationalizable policies are max-min solutions, which would then imply that the max-min concept has more bite than rationalizability in our context.

Proposition 7 *The max-min value of all possible (mixed strategy) subsidy policies is zero, i.e.,*

$$\max_{\sigma' \in \Delta(\Sigma_A)} \min_{\mu \in \Delta(\Sigma)} \int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma'(\mathbf{a})\, d\mu(p) = 0.$$

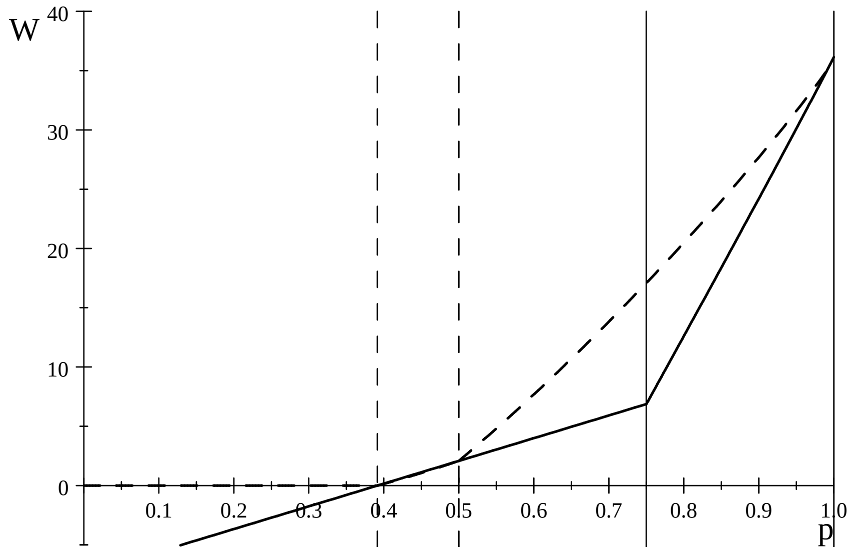

Figure 2: Solid curve: $W_1(p_0|\mathbf{a}_z)$ where $\mathbf{a}_z = (z, \alpha z, 0)$. Dashed curve: $W_1(p_0|\hat{\sigma})$, where $\hat{\sigma}$ is a mixed strategy over rationalizable matching subsidies.

The set of all pure-strategy max-min solutions is $A_{\max\min} \triangleq \left\{(0,0,\phi)\,|\phi \in \left[0, \phi_\gamma^{**}\right]\right\}$. *The set of all mixed-strategy max-min solutions is* $\{\sigma \in \Delta(\Sigma_A)\,|\sigma$ *is a mixed strategy over support* $A_{\max\min}\}$.

That is, the set of pure strategy max-min subsidy policies are those that involve no minimum mandate or unrestricted component and have matching rates ranging from 0 (no subsidy whatsoever) up to ϕ_γ^{**}. Furthermore, the pure matching policy $\left(0,0,\phi_\gamma^{**}\right)$ that induces neither over- nor underinvestment is optimal under the max-min criterion.

To contrast this set of max-min policies with the set of rationalizable policies, consider the policy $\mathbf{a}_z \triangleq (z, \alpha z, 0)$ for some small $z > 0$. From the expression for $W_1(p_0|\mathbf{a})$ in (19), it is clear that $\min_{\mu\in\Delta(\Sigma)} \int_0^1 W_1(p|\mathbf{a}_z)\, d\mu(p) < 0$, and hence $\mathbf{a}_z$ is not a max-min solution. However, we can find numerical examples such that $\mathbf{a}_z$ cannot be strictly dominated by any mixed strategy over support $A_{\max\min}$. Here we present one numerical example: let parameter values be $\alpha = 1$, $\lambda = 0.1$, $r = 0.01$, $\Pi = 20$, $CS = 30$, $\gamma = 0.5$, and set $z = 0.05$. In this case, we have $\phi_\gamma^{**} = 0.5$, $p_\gamma^{**} = 0.25$, and $p_1(0,0,0) = 0.5 \in \left(p_\gamma^{**}, 1\right)$.

In Figure 2 the solid curve is $W_1(p_0|\mathbf{a}_z)$ and the dashed curve represents a mixed strategy $\widehat{\sigma}$ over the support $\left\{(0,0,\phi)\,|\phi \in \left[0, \phi_\gamma^{**}\right]\right\}$. Letting $W_1(p_0|\widehat{\sigma}) \triangleq \int_{\mathbf{a}\in\left\{(0,0,\phi)|\phi\in\left[0,\phi_\gamma^{**}\right]\right\}} W_1(p|\mathbf{a})\, d\widehat{\sigma}(\mathbf{a})$, for $\widehat{\sigma}$ to strictly dominate $\mathbf{a}_z$, a necessary condition is that $W_1(p_0|\widehat{\sigma}) > W_1(p_0|\mathbf{a}_z)$ for $p_0 = 1$ and for all

$p_0 \in \left[\underline{p}(\mathbf{a}_z), p_1(0,0,0)\right]$, where $\underline{p}(\mathbf{a}_z)$ is defined such that $W_1\left(\underline{p}(\mathbf{a}_z)|\mathbf{a}_z\right) \equiv 0$. Numerically, $\underline{p}(\mathbf{a}_z) = 0.391\,3 \in \left(p_\gamma^{**}, p_1(0,0,0)\right)$. This example shows that if the constraint $W_1(p_0|\widehat{\sigma}) \geq W_1(p|\mathbf{a}_z)$ is imposed for $p \in \left\{\underline{p}(\mathbf{a}_z), p_1(0,0,0)\right\}$ then we must have $W_1(p|\widehat{\sigma}) < W_1(p|\mathbf{a}_z)$ for $p = 1$ for all $\widehat{\sigma}$ in the specified class of mixed strategies. This example reveals that the advantage of policy $\mathbf{a}_z$ over (pure strategy) matching subsidies is that the former performs well for both (extreme high type) $p_0 = 1$ and (low types) $p_0 \in \left[\underline{p}(\mathbf{a}_z), p_1(0,0,0)\right]$ (the interval between the two vertical dashed lines in Figure 2) at the same time, while a (pure strategy) matching subsidy can perform well for only one of the two scenarios at a time. Allowing mixed strategy matching subsidies can improve the performance of matching subsidy, but a trade off has to be made regarding how much probability weight to give to the matching subsidy that performs well for each scenario. Nevertheless, no mixed strategy matching subsidy can outperform the policy $\mathbf{a}_z$ for both $p_0 = 1$ and $p_0 \in \left\{\underline{p}(\mathbf{a}_z), p_1(0,0,0)\right\}$.

This example augments the intuition presented earlier for why an earmarked subsidy such as $\mathbf{a}_z$ would potentially be appealing to the government. For small $z > 0$, the investment by the extreme high-type firm, $p_0 = 1$, is not distorted, while the incurred shadow cost of public funds would be small, and the breakthrough would be likely to occur relatively early and end the subsidy. Meanwhile the low types $p_0 \in \left[\underline{p}(\mathbf{a}_z), p_1(0,0,0)\right]$ who would not invest without subsidy would be induced to invest. However, this appeal may be offset by the unappealing property of policy $\mathbf{a}_z$ that for the lower types $p_0 \in \left[0, \underline{p}(\mathbf{a}_z)\right)$, we have $W_1(p_0|\mathbf{a}_z) < 0$.

This now leads to a reason for justifying a preference of the (stronger) max-min criterion to rationalizability. For $\mathbf{a}_{s>0} = (z, s, \phi) \in A$ such that $s > 0$, we have $\min_{\mu \in \Delta(\Sigma)} \int_0^1 W_1(p|\mathbf{a}_{s>0})\, d\mu(p) < 0$. Moreover, for all $p_0 < p_\gamma^{**}$ we have $W_1(p_0|\mathbf{a}_{s>0}) < 0$. The danger of a policy such as $\mathbf{a}_{s>0}$ is that it attracts investment by firms of type $p_0 < p_\gamma^{**}$—that is , firms with sufficiently pessimistic beliefs about the viability of the project—which should not be induced to invest at all. The rationalizability criterion may be too weak to eliminate such a policy while the max-min criterion can with certainty.

For this reason, our preferred criterion is max-min. The key policy implications that follow from Proposition 7 are threefold:

- *Under the max-min criterion, every optimal subsidy policy is a pure matching policy. A subsidy policy with a minimum mandate or unrestricted component cannot satisfy the max-min criterion.*

- *Second, the matching rate under max-min policies ranges from 0 to* ϕ_γ^{**} *.*

- *Third, there exists a unique optimal subsidy policy under max-min criterion that induces neither underinvestment nor overinvestment in R&D, which is* $\left(0, 0, \phi_{\gamma}^{**}\right)$.

One important appeal of the pure matching policy is that it provides a screening device for the different types of firm to self-select, allowing crucial private information to be utilized for social benefit. Policies with $s > 0$ compromises this self-selection mechanism and wastes the valuable private information.

4.2 Optimal R&D Policy Under Incomplete Information: $N > 1$

With N firms, the first-best R&D investment policy is a straightforward extension of the first-best solution for the monopoly case:[23]

$$k^*(p) = \begin{cases} 1 & \text{if } p \in [p^*, 1] \\ 0 & \text{if } p \in [0, p^*] \end{cases},$$

where

$$p^* = \frac{\alpha}{\lambda[CS + \Pi + (N-1)\theta\Pi]} < 1.$$

To explore what welfare can actually be attained, as in the case of $N = 1$ we define the conditional expected social welfare function $W_N(p_0|\mathbf{a})$, which tells us the government's expected social welfare if it offered policy $\mathbf{a}$ and *it knew the firm's prior belief* p_0. Unlike the case of $N = 1$, $W_N(p_0|\mathbf{a})$ does not have a closed-form solution. In the Appendix, we characterize how it is determined.

As before, it is useful to consider the optimization program

$$\max_{\mathbf{a} \in A} W_N(p_0|\mathbf{a}). \tag{25}$$

An *ex post* equilibrium in a belief-free incomplete information game can be attained if there is a solution to this problem that is independent of the prior p_0. We now establish that when there is no shadow cost of public funds ($\gamma = 0$), the first-best outcome can be attained as an *ex post* equilibrium.

[23] It is straightforward to transform this problem to one that is identical to the first-best problem under monopoly using the transformations $\alpha' = N\alpha$, $\lambda' = N\lambda$, and $\Pi' = \Pi + (N-1)\theta\Pi$.

Proposition 8 *With $N > 1$ identical firms engaging in R&D, if a subsidy is a pure transfer between consumers and firms, i.e., $\gamma = 0$, then the following subsidy policy solves the maximization problem in (25) and thus constitutes an ex post equilibrium:*

$$\mathbf{a}^{**}(p_0) = \mathbf{a}^{**} = (0, r\theta\Pi, \frac{CS + N\theta\Pi}{CS + \Pi + (N-1)\theta\Pi}) \text{ for all } p_0 \in [0,1].$$

This policy induces the firm to choose an investment policy $k_N(p) = k^*(p)$ and thus achieves the first-best level of ex ante welfare for any prior belief p_0. If there are no spillovers ($\theta = 0$), the optimal subsidy policy for N firms is identical to that for a monopoly.

In the absence of a shadow cost of public funds, the subsidy policy that implements the first-best solution has an intuitively appealing form. The firm receives an unrestricted subsidy s that equals the flow equivalent of the spillover benefits $\theta\Pi$ that it would have received had another firm won the R&D competition. This ensures that the only way that a firm can improve its payoff is by winning the R&D competition, thus eliminating the free-rider problem and making the firms focus on winning the competition. Though a positive value of s eliminates the free-rider problem, it does not fully align the private marginal benefit of R&D with the social marginal benefit of R&D. By choosing the matching rate ϕ to equal the fraction of social surplus $\frac{CS+N\theta\Pi}{CS+\Pi+(N-1)\theta\Pi}$ that is not internalized, private and social incentives are aligned.

Does the result in Proposition 8 extend to the case when there is positive shadow cost of public funds? Again as in the case of single-firm R&D, the answer is no: an *ex post* equilibrium does not exist for positive value of γ. To substantiate this claim, as well as to facilitate the analysis of the case of $\gamma > 0$ and $N > 1$, we start by looking for a pure matching policy $\left(0, 0, \phi_{\gamma N}^{**}\right)$ that overcomes the possibilities of both overinvestment and underinvestment. Under such a policy, each firm would terminate investment at the threshold value:

$$p_N\left(0, 0, \phi_{\gamma N}^{**}\right) = p_{\gamma N}^{**} \triangleq \frac{\alpha\left(1 - \phi_{\gamma N}^{**}\right)}{\lambda\Pi},$$

(which is the point of both zero flow payoff and zero expected value for each firm), and the conditional government flow payoff and conditional welfare value become zero at $p = p_{\gamma N}^{**}$. Algebraically this implies:

$$\left\{\lambda p_{\gamma N}^{**}\left[CS + \Pi + (N-1)\theta\Pi\right] - \alpha - \gamma\phi_{\gamma N}^{**}\alpha\right\} N k_N\left(p_{\gamma N}^{**}\right) = 0,$$

which has the following solution and implication:

$$\phi_{\gamma N}^{**} = \frac{\frac{CS+(N-1)\theta\Pi}{1+\gamma}}{\Pi + \frac{CS+(N-1)\theta\Pi}{1+\gamma}},$$

$$p_{\gamma N}^{**} = \frac{\alpha}{\lambda\left[\Pi + \frac{CS+(N-1)\theta\Pi}{1+\gamma}\right]}.$$

Having defined $p_{\gamma N}^{**}$, we can now establish that an *ex post* equilibrium does not exist when $\gamma > 0$.

Proposition 9 *If $p_N(0,0,0) < q_N(0,0,0) < 1$, then for $\gamma > 0$ there exists no ex post equilibrium.*[24]

The intuition for this result is essentially the same as that for Proposition 4 in the case of a monopoly R&D. That is, the trade-off between the more R&D investment induced by a subsidy and the higher social cost resulting from the subsidy depends on the value of p_0 that is unknown to the government.

4.2.1 What Policy Should the Government Choose When $\gamma > 0$ and Multiple Firms Engage in R&D?

In the analysis of single-firm R&D with positive value of γ, we have argued for adopting the max-min criterion. The reasons employed in that argument can also be applied to the case of $N > 1$. Under this criterion, we can establish that $s = z = 0$ is entailed.

Proposition 10 *Under the max-min decision criterion, the optimal policy is a pure matching subsidy $(0,0,\phi)$ such that $\phi \in \left[0, \phi_{\gamma N}^{**}\right]$. The abandonment threshold is $p_N(0,0,\phi) = \frac{\alpha(1-\phi)}{\lambda\Pi}$.*

Under the max-min decision criterion, if the optimal policy has a further emphasis on overcoming the possibility of underinvestment, the best policy would be $\left(0,0,\phi_{\gamma N}^{**}\right)$. This policy induces investment to terminate at $p = p_{\gamma}^{**}$ which is socially optimal. However, since $(z,s) = (0,0)$, the threshold $\bar{\theta}(z,s) = 0 \leq \theta$, this policy cannot overcome the inter-firm free-rider problem, and it therefore cannot avoid underinvestment in terms of investment intensity; i.e., investment is not "flat-out" for $p \in \left(p_{\gamma N}^{**}, q_N\left(0,0,\phi_{\gamma N}^{**}\right)\right)$.

[24] The restrictive condition: $p_N(0,0,0) < q_N(0,0,0) < 1$ is imposed for the convenience of an analytical proof. Numerical examples show it is not a necessary condition for the result to hold.

4.2.2 R&D Consortium

One approach to overcoming the free-rider problem is to allow firms to cooperate by means of an N-firm research consortium. The research consortium serves as a vehicle for coordinating the investment decision of each individual firm, so that each firm in the consortium, when faced with subsidy policy $\mathbf{a}$, solves the following problem:

$$V(p) = \max_{k\in[z,1]} \left[(s - \alpha\phi z - \alpha(1-\phi)k)dt + \lambda N kpdt \left(\frac{\Pi + (N-1)\theta\Pi}{N} \right) + (1 - \lambda N kpdt) e^{-rdt} V(p+dp) \right].$$

In contrast to the case of $N = 1$, each firm in the consortium receives $\frac{1}{N}$ of the total private benefit $\Pi + (N-1)\theta\Pi$. Moreover, because all firms in the consortium simultaneously choose their individual investment to be k, each firm recognizes that the probability that a breakthrough will take place within $[t, t+dt)$ is $\lambda N kpdt$.

This can be transformed into the following differential equation

$$rV(p) = s - \alpha\phi z + \max_{k\in[z,1]} \left\{ -\alpha(1-\phi)k + \lambda N kp \left[\left(\frac{\Pi + (N-1)\theta\Pi}{N} \right) - V(p) - (1-p)V'(p) \right] \right\}.$$

Like a monopolist, each firm in the consortium has an optimal R&D policy $k_N^C(p)$ that is bang-bang:

$$k_N^C(p) = \begin{cases} 1 & \text{if } p > p_N^C(\mathbf{a}) \\ z & \text{if } p \le p_N^C(\mathbf{a}) \end{cases},$$

where the abandonment threshold $p_N^C(\mathbf{a})$ is given by

$$p_N^C(\mathbf{a}) = \frac{\alpha(1-\phi)}{\lambda \left[\Pi + (N-1)\theta\Pi - \frac{s-\alpha z}{r} \right] \left(\frac{r}{r+\lambda z} \right)}.$$

As before, we consider the optimization program

$$\max_{\mathbf{a}\in A} W_N^C(p_0|\mathbf{a}), \tag{26}$$

where $W_N^C(p_0|\mathbf{a})$ is the conditional expected social welfare function induced by the research consortium's optimal investment plan. This welfare $W_N^C(p_0|\mathbf{a})$ is fully analogous to $W_1(p_0|\mathbf{a})$, so we omit its expression for brevity. An *ex post* equilibrium in a belief-free incomplete information game can be attained if there is a solution to this problem that is independent of the prior p_0. We now establish that when there is no shadow cost of public funds ($\gamma = 0$), the first-best outcome can be attained as an *ex post* equilibrium.[25]

Proposition 11 *When N firms engage in cooperative R&D through a research consortium, if a subsidy is a pure transfer between consumers and firms, i.e., $\gamma = 0$, then the following subsidy policy solves the maximization problem in (26) and thus constitutes an ex post equilibrium: $(z^{**}(p_0), s^{**}(p_0), \phi^{**}(p_0)) = (z^{**}, s^{**}, \phi^{**}) = (0, 0, 1-\rho)$ for all $p_0 \in [0,1]$ (where recall $\rho = \frac{\Pi+(N-1)\theta\Pi}{\Pi+(N-1)\theta\Pi+CS}$). This policy induces the consortium to choose an investment policy $k_N^C(p) = k^*(p)$ and thus achieves the first-best level of ex ante welfare for any prior belief p_0.*

When there is a zero shadow cost of public funds, the government is indifferent between subsidizing N non-cooperative firms or N firms organized into a research consortium, provided that the matching rates and unrestricted subsidies are appropriately chosen as indicated in Propositions 8 and 11.

We note from Proposition 11 that when $\theta > 0$, the matching rate $\frac{CS}{CS+\Pi+(N-1)\theta\Pi}$ needed to attain the first-best outcome under a research consortium is less than the matching rate $\frac{CS+N\theta\Pi}{CS+\Pi+(N-1)\theta\Pi}$ needed to attain the first-best outcome under non-cooperative research. Further, with a research consortium attaining the first-best outcome entails no unrestricted subsidy, while an unrestricted subsidy is required under non-cooperative research. Thus, when there is no shadow cost of public funds, attaining the first-best outcome with a research consortium involves a smaller overall subsidy than attaining the first-best outcome with N non-cooperative firms. This suggests that the research consortium may have an advantage over non-cooperative research when the shadow cost of public funds is positive.

If there is a positive shadow cost of public funds, the analysis of the policy toward the research consortium is similar to the case of monopoly. Our preferred decision criterion therefore will also be the max-min criterion.

Proposition 12 *Under a research consortium, if $\gamma > 0$ and the max-min decision criterion is adopted, then the optimal subsidy policy is a pure matching subsidy $(0, 0, \phi)$ such that $\phi \in \left[0, \frac{\frac{CS}{1+\gamma}}{\frac{CS}{1+\gamma}+\Pi+(N-1)\theta\Pi}\right]$. The abandonment threshold is $p_N^C(0,0,\phi) = \frac{\alpha(1-\phi)}{\lambda[\Pi+(N-1)\theta\Pi]}$.*

[25] Because the proof of Proposition 11 is directly analogous to the proof of Proposition 3 in the Appendix and is thus omitted.

Under a research consortium and the max-min decision criterion by the government agency, if the optimal policy has a further emphasis on overcoming possible underinvestment, then the optimal policy should be $\left(0, 0, \frac{\frac{CS}{1+\gamma}}{\frac{CS}{1+\gamma}+\Pi+(N-1)\theta\Pi}\right)$, and this policy can overcome the possibility of underinvestment (both in terms of cumulative investment and investment intensity at each point of time). The lowest abandonment threshold is

$$p = \frac{\alpha\left(1 - \frac{\frac{CS}{1+\gamma}}{\frac{CS}{1+\gamma}+\Pi+(N-1)\theta\Pi}\right)}{\lambda\left[\Pi + (N-1)\theta\Pi\right]} = \frac{\alpha}{\lambda\left[\frac{CS}{1+\gamma} + \Pi + (N-1)\theta\Pi\right]} < p^{**}_{\gamma N} = \frac{\alpha}{\lambda\left[\Pi + \frac{CS+(N-1)\theta\Pi}{1+\gamma}\right]}.$$

The investment under the research consortium is thus higher than under non-cooperative research both in terms of cumulative investment and investment intensity. The inter-firm free-rider problem worsens the non-cooperative firms' incentive for investing which cannot be adequately compensated by pure matching subsidy policy. A research consortium has the potential to overcome this problem. To bring this point into sharp relief, note that for the same pure matching subsidy policy $(0, 0, \phi)$, we have

$$p^C_N\left(0, 0, \phi\right) = \frac{\alpha(1-\phi)}{\lambda\left[\Pi + (N-1)\,\theta\Pi\right]} < p_N\left(0, 0, \phi\right) = \frac{\alpha\left(1-\phi\right)}{\lambda\Pi},$$

which confirms that research consortium can help lower the abandonment threshold value.

5 Conclusions

In this paper, we study the optimal subsidy policy for research programs when the firm is privately informed about project viability and the government is unable to form a unique prior belief about the firm's private information. We first showed that different subsidy tools affect the firm's R&D incentives in different ways. In the case of monopoly R&D, a matching subsidy can stimulate R&D activity, while earmarked and unrestricted subsidies can suppress R&D activity. In the case of R&D competition, the incentive effects of subsidies are somewhat more complex. As in the case of monopoly R&D, an increase (*ceteris paribus*) in the matching rate expands the range over which firms invest in excess of the mandated minimum, while increases in the baseline unrestricted subsidy have the opposite effect. However, R&D competition involves the possibility of free riding (translating into a range in which, in equilibrium, firms' investment intensity is greater than the mandated minimum but less than the maximum feasible amount), and increases in both the mandated minimum and the unrestricted component of the subsidy can mitigate the extent of free riding by the firm.

We then studied the government's choice of an optimal subsidy policy. If there is no shadow cost of public funding, we show that the government can attain the first-best welfare outcome as an *ex post* (belief-free) equilibrium through the use of very simple subsidy policies. In the case of monopoly, the optimal subsidy scheme is a pure matching policy with the matching rate equal to the ratio of the portion of social welfare that is not appropriable by the firm to total social welfare. In the case of competition, the optimal policy is a combination of two instruments: a matching subsidy and an unrestricted subsidy. The unrestricted subsidy eliminates firms' incentives to free ride, and the matching subsidy solves the underinvestment due to the appropriability problem. Together, they ensure firms to follow the first-best investment path.

An *ex post* equilibrium does not exist when the shadow cost of public funding is positive. This necessitates consideration of what the government's criteria should be in selecting an optimal policy. We consider two criteria: rationalizability and max-min. We show that for both $N = 1$ and $N > 1$, the set of max-min policies consists entirely of pure matching subsidies. When $N = 1$, the policy with the highest matching rate solves the problem of underinvestment without inducing overinvestment. This is not the case when $N > 1$, since a pure matching subsidy is unable to correct for underinvestment that arises due to the free-rider problem. However, the highest max-min matching rate for a research consortium is less than the highest max-min matching rate under R&D competition, so allowing firms to determine R&D levels cooperatively can economize on the total shadow cost of the subsidy policy.

There are a number of interesting issues not addressed in this paper that warrant further attention. First, the belief updating structure in our paper is rather simple. As more time passes without a discovery, the updated likelihood that the project is viable falls. The simplicity of this updating rule allows us to derive a closed-form solution to our problem. However, this rule does not allow upward revision of the viability probability. To model this, we need to allow for the possibility that firms acquire new information as the research program progresses. This would be a useful extension of our model.

Second, as noted above, our paper restricts attention to time-invariant subsidy policies. In particular, this rules out the use of funding deadlines and terminations as a way of motivating R&D investment. Bonatti and Hörner (2011) study the use of deadlines in addressing the free-riding problem in an R&D collaboration. They show that a finite deadline T can be chosen to induce the agents to contribute maximum effort throughout the process. The intuition is that because agents cannot continue the project on their own once the deadline is reached, as the deadline approaches agents will increase their effort level to race against time. They show that there is an optimal time $T' < T$ after which the agents will exert maximum effort levels. Under certain conditions, T can be chosen so that $T' \leq 0$, which implies that the agents will exert full effort throughout the collaboration process.

Our paper differs from Bonatti and Hörner in that private incentives in our model are not only affected by the free-rider problem, but also by the appropriability problem. While Bonatti and Hörner show that a deadline can neutralize the free-rider problem, it is less clear that a deadline would fully neutralize the appropriability problem. Given the potential consumer benefit from the project that cannot be appropriated by firms, a social planner may not want to set a deadline such that firms are forbidden to conduct research after the deadline expires. Further, in the context of our model, government could not completely forbid firms to conduct self-funded research, and so a strict deadline on research activity would not be feasible. Still, deadlines on subsidies may be useful in our model. Removing governmental support after certain point in time would, as in Bonatti and Hörner, generate additional incentives. However, we conjecture that in contrast to the Bonatti and Hörner model, a subsidy deadline by itself would not achieve the first-best outcome when the appropriability problem is present. More generally, though, allowing for the possibility of a time-varying subsidy mechanism may move the social planner closer to the first-best solution in those cases in which time-invariant policies cannot attain the first-best solution (i.e., when there is a shadow-cost of public funds). In such cases, it would be useful to explore the interaction of deadlines and other subsidy instruments and in particular, whether a deadline is a complement to the matching rate, or a substitute for it, in generating incentives.

6 Appendix

Proof of Proposition 1:

Because the objective function is linear in k, the firm's optimal investment decision $k_1(p)$ is a "bang-bang" rule. This implies that it either sets k equal to the minimum required level z, or its maximum feasible level 1, i.e.,

$$k_1(p) = \begin{cases} 1 & \text{if } p > p_1 \\ z & \text{if } p \leq p_1 \end{cases} .$$

If $k = z$, the general solution to the equation (6) is [26]

$$V_L(p) = \frac{s - \alpha z}{r} + \frac{\lambda z}{r + \lambda z}\frac{r\Pi - (s - \alpha z)}{r}p.$$

If $k = 1$, the general solution to the equation (6) is

$$V_H(p) = \frac{s - (1-\phi)\alpha - \alpha\phi z}{r} + \frac{\lambda}{r+\lambda}\left(\Pi - \left[\frac{s - (1-\phi)\alpha - \alpha\phi z}{r}\right]\right)p + B_1 p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}},$$

where B_1 is a constant. The abandonment threshold p_1 and the constant B_1 are determined by the value matching and smooth pasting conditions for $V_L(p)$ and $V_H(p)$:

$$V_L(p_1) = V_H(p_1). \tag{27}$$

$$V_L'(p_1) = V_H'(p_1). \tag{28}$$

Algebraic analysis of (27) and (28) reveal that p_1 can be expressed as the solution to (7) above, and the p_1 and B_1 that solve (27) and (28) are

$$p_1(\mathbf{a}) = \frac{\alpha(1-\phi)(r+\lambda z)}{\lambda(r\Pi - s + \alpha z)}.$$

$$B_1(\mathbf{a}) = \frac{\lambda\alpha(1-z)(1-\phi)}{r(r+\lambda)}\left(\frac{1-p_1}{p_1}\right)^{-\frac{r}{\lambda}}.$$

[26] Note that there is a nonlinear term $B_0 p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda z}{\lambda z}}$ to the general soluction of this differential equation, but is dropped because $V(0)$ needs to be finite and thus implies $B_0 = 0$. In fact, $V(0)$ represents the value of the firm when the R&D project is destined to fail.

Thus,

$$V_1(p) = \begin{cases} V_H(p) = \frac{s-(1-\phi)\alpha-\alpha\phi z}{r} + \frac{\lambda}{r+\lambda}\left(\Pi - \left[\frac{s-(1-\phi)\alpha-\alpha\phi z}{r}\right]\right)p + B_1(\mathbf{a})p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}} & \text{if } p > p_1(\mathbf{a}) \\ V_L(p) = \frac{s-\alpha z}{r} + \frac{\lambda z}{r+\lambda z}\frac{r\Pi-(s-\alpha z)}{r}p & \text{if } p \leq p_1(\mathbf{a}) \end{cases}.$$

■

Proof of Proposition 2:[27]

We begin by defining

$$b\left(p, V^i\right) = \lambda p\left[\theta\Pi - V^i - (1-p)V^{i\prime}\right]$$
$$c(p) = \alpha(1-\phi) - \lambda p(1-\theta)\Pi.$$

The term $b\left(p, V^i\right)$ is the marginal benefit to the firm from an additional unit of investment effort by a rival firm, while $c(p)$ is the *net* marginal cost to the firm from an additional unit of R&D. We can write the Bellman equation (10) as

$$rV^i(p) - (s - \alpha\phi z) = K^{-i}b\left(p, V^i\right) + \max_{k^i\in[z,1]}\left\{k^i\left[b\left(p, V^i\right) - c(p)\right]\right\},$$

The firm's optimal investment decision is given by the following "reaction function":

$$k^i(p) = \begin{cases} 1 & \text{if } V^i(p) > \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r}K^{-i} \\ \in [z,1] & \text{if } V^i(p) = \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r}K^{-i} \\ z & \text{if } V^i(p) < \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r}K^{-i} \end{cases}. \quad (29)$$

The linearity of the value function implies that we only need to consider three cases: $k = z$, $k = 1$ and $k = \kappa \in (z,1)$. As noted, we seek to characterize a symmetric equilibrium with investment policies $k^i(p) = k_N(p)$ and value function $V^i(p) = V_N(p)$, for all $i = 1, ..., N$. When $k_N(p) = z$, the differential

[27] Part (i) of the proof follows the method employed by Keller Rady and Cripps (2005).

equation (10) is:

$$rV_N(p) = s - \alpha\phi z + \lambda p(N-1)z\left[\theta\Pi - V_N(p) - (1-p)V_N'(p)\right]$$
$$+ z\left[\lambda p\left[\Pi - V_N(p) - (1-p)V_N'(p)\right] - \alpha(1-\phi)\right].$$

The solution to this equation, denoted by $V_N^L(p)$, is:[28]

$$V_N^L(p) = \frac{s-\alpha z}{r} + \frac{\lambda Nz\left[\frac{r(1-\theta)\Pi}{N} + r\theta\Pi - s + \alpha z\right]}{r(r+\lambda Nz)}p.$$

When $k_N(p) = 1$, the differential equation (10) is

$$rV_N(p) = s - \alpha\phi z + \lambda p(N-1)\left[\theta\Pi - V_N(p) - (1-p)V_N'(p)\right]$$
$$+ \left[\lambda p\left[\Pi - V_N(p) - (1-p)V_N(p)\right] - \alpha(1-\phi)\right].$$

The solution to this equation, denoted by $V_N^H(p)$, is

$$V_N^H(p) = \frac{s-\alpha(1-\phi)-\alpha\phi z}{r} + \frac{\lambda N\left(\frac{r(1-\theta)\Pi}{N} + r\theta\Pi - s + \alpha(1-\phi) + \alpha\phi z\right)}{r(r+\lambda N)}p + B_H p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda N}{\lambda N}},$$

where B_H is a constant to be determined. Finally, if $k_N(p) \in (z,1)$, then from (29), $b(V_N(p), p) = c(p)$, or

$$\lambda p\left(\theta\Pi - V_N(p) - (1-p)V_N'(p)\right) - \left[\alpha(1-\phi) - \lambda p(1-\theta)\Pi\right] = 0.$$

The solution to this equation, which is denoted by $V_N^M(p)$, is

$$V_N^M(p) \equiv \frac{\lambda\Pi - (1-\phi)\alpha}{\lambda} - B_M(1-p) + \frac{(1-\phi)\alpha(1-p)}{\lambda}\ln\frac{1-p}{p}, \tag{30}$$

where B_M is a constant to be determined. The value matching and smooth pasting conditions imply

[28] The coefficient to the nonlinear part is zero as we require finiteness for the value function at $p = 0$.

that

$$V_N^M(p_N) = V_N^L(p_N)$$
$$V_N^{M\prime}(p_N) = V_N^{L\prime}(p_N),$$

which gives us

$$p_N(\mathbf{a}) = \frac{\alpha(1-\phi)}{\lambda\left(\left[\Pi - \frac{(s-\alpha z)}{r}\right]\left(\frac{r}{r+\lambda N z}\right) + (1-\theta)\Pi\left(\frac{(N-1)\lambda z}{r+\lambda N z}\right)\right)}. \tag{31}$$

Now, from (29), $k_N(p) \in (z, 1)$ if and only if $V_N^M(p) = \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r}(N-1)k_N(p)$. This implies:

$$k_N(p) = \frac{rV_N^M(p) - s + \alpha\phi z}{(N-1)c(p)}, \tag{32}$$

which makes each firm indifferent between choosing any investment level between z and 1. A necessary condition for this to be well defined is that $c(p_N)$ be positive; otherwise $k_N(p)$ will be negative. From the definition of $c(p)$ above, $c(p_N) > 0$ if and only if $p_N < \frac{(1-\phi)\alpha}{\lambda(1-\theta)\Pi}$. Straightforward algebra establishes that this condition is equivalent to

$$\theta > \overline{\theta}(z, s) \triangleq 1 - \frac{\Pi - \left(\frac{s-\alpha z}{r}\right)}{\Pi}\frac{r}{r+\lambda z}.$$

Since $k_N(p) \to \infty$ as $p \to \frac{(1-\phi)\alpha}{\lambda(1-\theta)\Pi}$, there exists $q_N(\mathbf{a})$ such that $k_N(q_N(\mathbf{a})) = 1$. Substituting (30) into (32) and equating to 1 yields condition (13) above. In this case, the value function is thus:

$$V_N(p) = \begin{cases} V_N^H(p) & \text{if } p \geq q_N(\mathbf{a}) \\ V_N^M(p) & \text{if } p_N(\mathbf{a}) < p < q_N(\mathbf{a}) \\ V_N^L(p) & \text{if } p \leq p_N(\mathbf{a}) \end{cases},$$

with $B_H(\mathbf{a})$ satisfying $V_H^H(q_N) = V_N^M(q_N)$, or equivalently, it equates (V_{N1}) and (V_{N2}).

(ii) If $\theta \leq \overline{\theta}(z, s)$, we will show that each firm's optimal strategy is either to invest "flat out" in R&D effort by setting $k = 1$ when $p > p_N(\mathbf{a})$ and invest at the minimum level z, when $p \leq p_N(\mathbf{a})$. To establish the latter, suppose all other firms besides i are investing the minimum level z, so that $K^{-i} = (N-1)z$. To show that firm i's best response is also to invest z when $p \leq p_N(\mathbf{a})$, then from

(29), we must establish that $V_N(p) < \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r}(N-1)K^{-i}$, or equivalently $V_N^L(p) < \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r}(N-1)z$, when $p \leq p_N(\mathbf{a})$. Using the expressions for $V_N^L(p)$ and $c(p)$, and some straightforward algebraic manipulations, $V_N^L(p) < \frac{s-\alpha\phi z}{r} + \frac{c(p)}{r}(N-1)z$ can be shown to be equivalent to $\lambda\left(\left[\Pi - \frac{(s-\alpha z)}{r}\right]\left(\frac{r}{r+\lambda N z}\right) + (1-\theta)\Pi\left(\frac{(N-1)\lambda z}{r+\lambda N z}\right)\right)p \leq \alpha(1-\phi)$, and given the expression for $p_N(\mathbf{a})$ in (31), this indeed is true for $p \leq p_N(\mathbf{a})$. Now assume $p > p_N(\mathbf{a})$ and every other firm invests $k=1$. We need to show that firm i's best response is to invest $k=1$. Suppose not. Then, due to the linearity of the problem, the only case that the firm will invest $z < k < 1$ is when $V^i = \frac{s}{r} + \frac{c(p)}{r}K^{-i}$, but this implies $p_N(\mathbf{a}) < \frac{(1-\phi)\alpha}{\lambda(1-\theta)\Pi}$. However, $\theta \leq \overline{\theta}(z,s)$ implies $z \geq \frac{r\theta\Pi - s}{\lambda(1-\theta)\Pi - \alpha}$, which, in turn, implies $p_N(\mathbf{a}) \geq \frac{(1-\phi)\alpha}{\lambda(1-\theta)\Pi}$, a contradiction. The uniqueness also follows from the fact that the investment problem is linear in k so firms will not coordinate in investing in a lower level $\kappa \in (z,1)$ because if that is the case, it then implies $b\left(p, V_i^n\right) > c\left(p\right)$, in which case the firm will invest $k=1$ rather than $\kappa < 1$.[29] In this case the value function is:

$$V_N(p) = \begin{cases} V_N^H\left(p\right) & \text{if } p > p_N(\mathbf{a}) \\ V_N^L\left(p\right) & \text{if } p \leq p_N(\mathbf{a}) \end{cases}.$$

■

Derivation of the welfare schedule $W_1(p)$:

As Proposition 1 shows, if $\frac{\alpha(1-\phi)(r+\lambda z)}{\lambda(r\Pi - s + \alpha z)} < 1$, the monopoly firm either invests $k=1$ or $k=z$ in the R&D project. If $k_1(p)=1$ (which occurs if $p > p_1(\mathbf{a})$), the solution to the differential equation in (17) is:

$$W_1\left(p\right) = -\left(\frac{\alpha + \gamma s + \gamma\phi\alpha(1-z)}{r}\right) + \frac{\lambda}{r+\lambda}\left[CS + \Pi + \left(\frac{\alpha + \gamma s + \gamma\phi\alpha(1-z)}{r}\right)\right]p + B_W p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}}.$$

If $k_1(p) = z$ (which occurs if $p < p_1$), the solution to the differential equation in (17) is:[30]

$$W_1\left(p\right) = -\left(\frac{\alpha z + \gamma s}{r}\right) + \frac{\lambda z}{r+\lambda z}\left[CS + \Pi + \left(\frac{\alpha z + \gamma s}{r}\right)\right]p.$$

[29] For a similar proof of this result, see Besanko and Wu (2013).

[30] As usual, we drop the nonlinear term by requiring the value function to be finite at $p = 0$.

Thus,

$$W_1(p) = \begin{cases} -\left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right) + \frac{\lambda}{r+\lambda}\left[CS+\Pi+\left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)\right]p + B_W p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda}{\lambda}} & \text{if } p > p_1(\mathbf{a}) \\ -\left(\frac{\alpha z+\gamma s}{r}\right) + \frac{\lambda z}{r+\lambda z}\left[CS+\Pi+\left(\frac{\alpha z+\gamma s}{r}\right)\right]p & \text{if } p \leq p_1(\mathbf{a}) \end{cases}, \tag{33}$$

Because at p_1, the firm is indifferent between $k=1$ and $k=z$, the welfare schedule is continuous at p_1. Thus, the constant B_W equates the upper piece of $W_1(p)$ and the lower piece. Straightforward algebra establishes

$$B_W = B_W(p_1,\mathbf{a}) = \frac{\left\{\begin{array}{c} \frac{\alpha(1-z)}{r} + \frac{\gamma\phi\alpha(1-z)}{r} + \\ \frac{\lambda}{r+\lambda}\left[CS+\Pi+\left(\frac{\alpha+\gamma s+\gamma\phi\alpha(1-z)}{r}\right)\right]p_1 \\ -\frac{\lambda z}{r+\lambda z}\left[CS+\Pi+\left(\frac{\alpha z+\gamma s}{r}\right)\right]p_1 \end{array}\right\}}{p_1\left(\frac{(1-p_1)}{p_1}\right)^{\frac{r+\lambda}{r}}}$$

If $\frac{\alpha(1-\phi)(r+\lambda z)}{\lambda(r\Pi-s+\alpha z)} \geq 1$, the monopoly firm invests $k=z$ in the R&D project for all p, and thus

$$W_1(p) = -\left(\frac{\alpha z+\gamma s}{r}\right) + \frac{\lambda z}{r+\lambda z}\left[CS+\Pi+\left(\frac{\alpha z+\gamma s}{r}\right)\right]p.$$

■

Proof of Proposition 3:

To prove the proposition, we will show that the first-best level of welfare can be attained by setting $s=z=0$ and choosing $\phi = 1-\rho$. The first-best R&D policy $k^*(p)$ solves the social planner's problem:

$$W^*(p) = \max_{k\in[0,1]}\left[-\alpha k dt + \lambda k p dt\,(CS+\Pi) + (1-\lambda k p dt)e^{-rdt}W^*(p+dp)\right]. \tag{34}$$

Using an approach similar for proving Proposition 1, the first-best R&D policy is given by:

$$k^*(p) = \begin{cases} 1 & \text{if } p > p^* \\ 0 & \text{if } p \leq p^* \end{cases}, \tag{35}$$

where

$$p^* = \frac{\alpha}{\lambda(\Pi+CS)}.^{31}$$

[31] Because we have assumed that the social benefit-cost ratio for a viable project exceeds 1, we have $\frac{\lambda(CS+\Pi)}{\alpha} > 1$, and

is the first-best abandonment threshold.

Now, when $\gamma = 0$,

$$W_1(p) = -\alpha k_1(p)dt + \lambda k_1(p)pdt\,(CS + \Pi) + (1 - \lambda k_1(p)pdt)e^{-rdt}W_1(p + dp)\,,$$

and because of (34), for any arbitrary subsidy policy, it must be the case that $W_1(p) \leq W^*(p)$, and in particular $W_1(p_0|\mathbf{a}) \leq W^*(p_0)$. However, if we can find a subsidy policy such that $k_1(p) = k^*(p)$,then $W_1(p) = W^*(p)$ and in particular, $W_1(p_0|\mathbf{a}) = W^*(p_0)$ for any prior belief p_0 the firm might have. A subsidy policy that implements the first-best investment policy must therefore maximize expected *ex ante* welfare when $\gamma = 0$.

Note that if $s = z = 0$,

$$p_1(\mathbf{a}) = p_1(0, 0, \phi) = \frac{\alpha(1-\phi)}{\lambda\,(\Pi + CS)}.$$

A matching rate given by $\phi = 1 - \frac{\Pi}{\Pi + CS} = 1 - \rho$, along with $s = z = 0$, ensures that $p_1(\mathbf{a}) = p^*$, and thus implements the maximum level of expected welfare for any prior belief p_0. Therefore, $z = 0, s = 0$, $\phi = 1 - \frac{\Pi}{\Pi + CS}$ is the optimal policy.■

Proof of Proposition 4:

Suppose the contrary, i.e., there exists a policy $\mathbf{a} \in A$ such that $\mathbf{a}$ does not depend on p_0, but maximizes $W_1(p_0|\mathbf{a})$ for all $p_0 \in [0, 1]$. Now, when $p_0 = 1$, by (19) we know

$$W_1(1|\mathbf{a}) = \left\{ \begin{array}{c} -\left(\frac{\alpha + \gamma s + \gamma\phi\alpha(1-z)}{r}\right) \\ +\frac{\lambda}{r+\lambda}\left[CS + \Pi + \left(\frac{\alpha + \gamma s + \gamma\phi\alpha(1-z)}{r}\right)\right] \end{array} \right\},$$

and in this case the policy $(0, 0, 0)$ can easily be shown to be uniquely optimal (provided $p_1^{NO} < 1$). Thus, it is necessary that $\mathbf{a} = (0, 0, 0)$. However, for $p_0 \in \left(\frac{\alpha}{\lambda\left(\Pi + \frac{CS}{1+\gamma}\right)}, p_1^{NO}\right)$, the policy $\mathbf{a}$ cannot induce investment and thus is not optimal. To see this, notice that

$$p_0\lambda\,(\Pi + CS) - \alpha - \gamma\alpha\frac{\frac{CS}{1+\gamma}}{\Pi + \frac{CS}{1+\gamma}} > 0 \text{ if } p_0 > \frac{\alpha}{\lambda\left(\Pi + \frac{CS}{1+\gamma}\right)},$$

thus $p^* < 1$. This implies that there must exist some set of prior beliefs for which flat-out R&D investment would occur under the socially optimal policy.

that is, the expected flow payoff from investment for the government under policy $\left(0, 0, \frac{\frac{CS}{1+\gamma}}{\Pi+\frac{CS}{1+\gamma}}\right)$ is strictly positive; and

$$p_0 \lambda \Pi - \left(1 - \frac{\frac{CS}{1+\gamma}}{\Pi + \frac{CS}{1+\gamma}}\right) \alpha > 0 \text{ if } p_0 > \frac{\alpha}{\lambda \left(\Pi + \frac{CS}{1+\gamma}\right)},$$

that is, the expected flow payoff from investment for the firm under policy $\left(0, 0, \frac{\frac{CS}{1+\gamma}}{\Pi+\frac{CS}{1+\gamma}}\right)$ is strictly positive. Therefore the policy $\left(0, 0, \frac{\frac{CS}{1+\gamma}}{\Pi+\frac{CS}{1+\gamma}}\right)$, which can induce investment, is superior to $\mathbf{a}$. This contradiction with the supposition that $\mathbf{a}$ is an *ex post* equilibrium can only imply that no *ex post* equilibrium exists.■

Proof of Proposition 5:

The "only if" part is obvious because if there exist σ'' that dominates σ, then we must have

$$\int_0^1 \int_{\mathbf{a} \in A} W_1 (p|\mathbf{a})\, d\sigma (\mathbf{a})\, d\mu (p) < \int_0^1 \int_{\mathbf{a} \in A} W_1 (p|\mathbf{a})\, d\sigma'' (\mathbf{a})\, d\mu(p) \leq \max_{\sigma' \in \Delta(\Sigma_A)} \int_0^1 \int_{\mathbf{a} \in A} W_1 (p|\mathbf{a})\, d\sigma' (a)\, d\mu (p),$$

which means σ is never a best response, i.e., not rationalizable.

For the "if" part, suppose σ is not rationalizable, that is, for any $\mu \in \Delta (\Sigma)$ there exists $b (\mu) \in \arg\max_{\sigma' \in \Delta(\Sigma_A)} \int_0^1 \int_{\mathbf{a} \in A} W_1 (p|\mathbf{a})\, d\sigma' (a)\, d\mu (p)$, such that

$$\int_0^1 \int_{\mathbf{a} \in A} W_1 (p|\mathbf{a})\, d\sigma (\mathbf{a})\, d\mu (p) < \int_0^1 \int_{\mathbf{a} \in A} W_1 (p|\mathbf{a})\, db (\mu)\, d\sigma (\mathbf{a})\, d\mu (p).$$

Define a two player zero-sum game such that $\left(\sigma^1, \mu^2\right) \in \Delta (\Sigma_A) \times \Delta (\Sigma)$ are a strategy profile and player 1's payoff is

$$U^1 \left(\sigma^1, \mu^2\right) = \int_0^1 \int_{\mathbf{a} \in A} W_1 (p|\mathbf{a})\, d\sigma^1 (\mathbf{a})\, d\mu^2 (p) - \int_0^1 \int_{\mathbf{a} \in A} W_1 (p|\mathbf{a})\, d\sigma (\mathbf{a})\, d\mu^2 (p),$$

and player 2's payoff is $U^2 \left(\sigma^1, \mu^2\right) = -U^1 \left(\sigma^1, \mu^2\right)$. By (the extension of) the von Neumann Minimax

Theorem[32], there exists $\left(\sigma^{1*}, \mu^{2*}\right)$ such that

$$\sigma^{1*} \in \arg \max_{\sigma^1 \in \Delta(\Sigma_A)} \min_{\mu^2 \in \Delta(\Sigma)} U^1\left(\sigma^1, \mu^2\right),$$

$$\mu^{2*} \in \arg \min_{\mu^2 \in \Delta(\Sigma)} \max_{\sigma^1 \in \Delta(\Sigma_A)} U^1\left(\sigma^1, \mu^2\right),$$

and

$$\max_{\sigma^1 \in \Delta(\Sigma_A)} \min_{\mu^2 \in \Delta(\Sigma)} U^1\left(\sigma^1, \mu^2\right) = U^1\left(\sigma^{1*}, \mu^{2*}\right) = \min_{\mu^2 \in \Delta(\Sigma)} \max_{\sigma^1 \in \Delta(\Sigma_A)} U^1\left(\sigma^1, \mu^2\right).$$

Furthermore, $\left(\sigma^{1*}, \mu^{2*}\right)$ is a (mixed strategy) Nash equilibrium. It follows

$$\begin{aligned} U^1\left(\sigma^{1*}, \mu^2\right) &\geq U^1\left(\sigma^{1*}, \mu^{2*}\right) \\ &\geq U^1\left(b\left(\mu^{2*}\right), \mu^{2*}\right) \\ &> U^1\left(\sigma, \mu^{2*}\right) \\ &= 0. \end{aligned}$$

The equality follows from the definition of $U^1\left(\sigma^1, \mu^2\right)$. But

$$U^1\left(\sigma^{1*}, \mu^2\right) > 0 \ \forall \mu^2 \in \Delta(\Sigma) \Rightarrow$$
$$\int_0^1 \int_{\mathbf{a} \in A} W_1\left(p|\mathbf{a}\right) d\sigma^{1*}\left(\mathbf{a}\right) d\mu^2\left(p\right) > \int_0^1 \int_{\mathbf{a} \in A} W_1\left(p|\mathbf{a}\right) d\sigma\left(\mathbf{a}\right) d\mu^2\left(p\right) \ \forall \mu^2 \in \Delta(\Sigma).$$

Thus, σ is strictly dominated by σ^{1*}.■

Proof of Proposition 6:

Suppose the opposite, that is, $\exists \sigma \in \Delta(\Sigma_A)$ such that σ is a max-min solution but not rationalizable. Then by Proposition 5, σ must be strictly dominated, i.e., $\exists \sigma' \in \Delta(\Sigma_A)$ such that

$$\int_0^1 \int_{\mathbf{a} \in A} W_1\left(p|\mathbf{a}\right) d\sigma'\left(\mathbf{a}\right) d\mu\left(p\right) > \int_0^1 \int_{\mathbf{a} \in A} W_1\left(p|\mathbf{a}\right) d\sigma\left(\mathbf{a}\right) d\mu\left(p\right) \text{ for all } \mu \in \Delta(\Sigma).$$

[32]The (original) von Neumann Minimax Theorem applies to finite two-person zero-sum games. Since the pure strategy spaces A and $[0,1]$ are both convex and compact subsets of Euclidian spaces, what we need is an extension to infinite games where the pure strategy spaces are convex compact sets in topological linear spaces, and the payoff function is continuous. The proof of such an extension has been given by Nikaidô (1954).

Consequently, for all $\mu \in \Delta(\Sigma)$,

$$\begin{aligned}
&\min_{\mu' \in \Delta(\Sigma)} \int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma(\mathbf{a})\, d\mu'(p) \\
&\leq \int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma(\mathbf{a})\, d\mu(p) \\
&< \int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma'(\mathbf{a})\, d\mu(p).
\end{aligned}$$

The universality of μ implies

$$\min_{\mu' \in \Delta(\Sigma)} \int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma(\mathbf{a})\, d\mu'(p) \leq \int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma(\mathbf{a})\, d\mu''(p) \text{ for any } \mu'' \in \Delta(\Sigma).$$

In particular, let $\mu'' \in \arg\min_{\mu \in \Delta(\Sigma)} \int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma'(\mathbf{a})\, d\mu(p)$, we have

$$\begin{aligned}
&\min_{\mu' \in \Delta(\Sigma)} \int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma(\mathbf{a})\, d\mu'(p) \\
&\leq \int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma(\mathbf{a})\, d\mu''(p) \\
&< \int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma'(\mathbf{a})\, d\mu''(p) \\
&= \min_{\mu \in \Delta(\Sigma)} \int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma'(\mathbf{a})\, d\mu(p).
\end{aligned}$$

The above strict inequality is a contradiction to the supposition that σ is a max-min solution since σ' has a higher minimum value.■

Proof of Proposition 7:

For $p_0 = 0$, it is obvious that $W_1(p_0|\mathbf{a}) \leq 0$ for all $\mathbf{a} \in A$ and $W_1(p_0|(0,0,0)) = 0$. Consequently,

$$\max_{\sigma' \in \Delta(\Sigma_A)} \min_{\mu \in \Delta(\Sigma)} \int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma'(a)\, d\mu(p) = 0.$$

Since for any $\mathbf{a} = (z, s, \phi) \in A$ such that $s > 0$,

$$W_1(0|\mathbf{a}) < 0 = \max_{\sigma' \in \Delta(\Sigma_A)} \min_{\mu \in \Delta(\Sigma)} \int_0^1 \int_{\mathbf{a} \in A} W_1(p|\mathbf{a})\, d\sigma'(\mathbf{a})\, d\mu(p),$$

$\mathbf{a}$ cannot be a max-min solution. That is, $s = 0$ is a necessary conditions for a max-min solution, and since we require $s \geq \alpha z$ and $z \geq 0$, $z = 0$ must also be a necessary condition as well.

Now, for any $\mathbf{a} = (0, 0, \phi) \in A$ such that $\phi > \phi^{**}_{\gamma}$, it can be verified that $W_1(p_0|(0,0,\phi)) < 0$ for

$p_0 \in \left[p_1\left((0,0,\phi)\right), p_\gamma^{**}\right)$. To verify this, it suffices to establish that for $\phi = \phi_\gamma^{**}$ from social point of view, p_γ^{**} is the optimal stopping point, which has continuation value of zero (for the government), and we have so argued in the main text above. It follows that for $\phi > \phi_\gamma^{**}$, which induces overinvestment and a higher total shadow cost of public funds, $W_1(p_0|(0,0,\phi)$ has negative value. Thus

$$W_1\left(p_0|\left(0,0,\phi\right)\right) < 0 = \max_{\sigma' \in \Delta(\Sigma_A)} \min_{\mu \in \Delta(\Sigma)} \int_0^1 \int_{\mathbf{a} \in A} W_1\left(p|\mathbf{a}\right) d\sigma'\left(\mathbf{a}\right) d\mu\left(p\right),$$

that is, $(0,0,\phi)$ cannot be a (pure strategy) max-min solution.

For all $\phi \in \left[0, \phi_\gamma^{**}\right]$,

$$\min_{\mu \in \Delta(\Sigma)} \int_0^1 W_1\left(p|\left(0,0,\phi\right)\right) d\mu\left(p\right) = 0,$$

that is, $(0,0,\phi)$ is a pure strategy max-min solution for all $\phi \in \left[0, \phi_\gamma^{**}\right]$.

For any mixed strategy σ which has positive probability mass over $\{(0,0,\phi')\,|\phi' \in [\phi,1]\}$ for some $\phi > \phi_\gamma^{**}$, we have

$$\begin{aligned}
&\min_{\mu \in \Delta(\Sigma)} \textstyle\int_0^1 \int_{\mathbf{a} \in \{(0,0,\phi')|\phi' \in [0,1]\}} W_1\left(p|\mathbf{a}\right) d\sigma\left(\mathbf{a}\right) d\mu\left(p\right) \\
&\leq \textstyle\int_0^1 \int_{\mathbf{a} \in \{(0,0,\phi')|\phi' \in [0,1]\}} W_1\left(p|\mathbf{a}\right) d\sigma\left(\mathbf{a}\right) d\delta_{p_0}\left(p\right), \text{ where } \delta_{p_0}\left(p\right) = \begin{cases} 1 & \text{for } p = p_0 \\ 0 & \text{for } p \neq p_0 \end{cases}, p_0 \in \left(p_1\left((0,0,\phi)\right), p_\gamma^{**}\right) \\
&= \textstyle\int_0^1 \int_{\mathbf{a} \in \{(0,0,\phi')|\phi' \in [\phi,1]\}} W_1\left(p_0|\mathbf{a}\right) d\sigma\left(\mathbf{a}\right) \\
&< 0 = \max_{\sigma' \in \Delta(\Sigma_A)} \min_{\mu \in \Delta(\Sigma)} \textstyle\int_0^1 \int_{\mathbf{a} \in A} W_1\left(p|\mathbf{a}\right) d\sigma'\left(\mathbf{a}\right) d\mu\left(p\right),
\end{aligned}$$

that is, σ cannot be a mixed strategy max-min solution.

Also, for any mixed strategy σ over support $A_{\max\min} \triangleq \left\{(0,0,\phi)\,|\phi \in \left[0, \phi_\gamma^{**}\right]\right\}$, we have

$$\begin{aligned}
&\min_{\mu \in \Delta(\Sigma)} \textstyle\int_0^1 \int_{\mathbf{a} \in \left\{(0,0,\phi')|\phi' \in \left[0,\phi_\gamma^{**}\right]\right\}} W_1\left(p|\mathbf{a}\right) d\sigma\left(a\right) d\mu\left(p\right) \\
&\leq \textstyle\int_0^1 \int_{\mathbf{a} \in \left\{(0,0,\phi')|\phi' \in \left[0,\phi_\gamma^{**}\right]\right\}} W_1\left(p|a\right) d\sigma\left(\mathbf{a}\right) d\delta_0\left(p\right) \\
&= \textstyle\int_0^1 \int_{\mathbf{a} \in \left\{(0,0,\phi')|\phi' \in \left[0,\phi_\gamma^{**}\right]\right\}} W_1\left(0|\mathbf{a}\right) d\sigma\left(\mathbf{a}\right), \\
&= 0.
\end{aligned}$$

where the next to last equality follows from the property of the Dirac measure $\delta_0\left(p\right)$. Now, since

$W_1(p|\mathbf{a}) \geq 0$ for all $p \in [0,1]$, all $\mathbf{a} \in \left\{(0,0,\phi')\,|\phi' \in \left[0,\phi_\gamma^{**}\right]\right\}$, we also have

$$\begin{aligned}
&\min_{\mu\in\Delta(\Sigma)} \textstyle\int_0^1 \int_{\mathbf{a}\in\left\{(0,0,\phi')|\phi'\in\left[0,\phi_\gamma^{**}\right]\right\}} W_1\left(p|\mathbf{a}\right) d\sigma\left(a\right) d\mu\left(p\right) \\
&\geq \min_{\mu\in\Delta(\Sigma)} \textstyle\int_0^1 \int_{\mathbf{a}\in\left\{(0,0,\phi')|\phi'\in\left[0,\phi_\gamma^{**}\right]\right\}} 0 d\sigma\left(\mathbf{a}\right) d\mu\left(p\right) \\
&= 0.
\end{aligned}$$

Overall, we must have

$$\min_{\mu\in\Delta(\Sigma)} \int_0^1 \int_{\mathbf{a}\in\left\{(0,0,\phi')|\phi'\in\left[0,\phi_\gamma^{**}\right]\right\}} W_1\left(p|\mathbf{a}\right) d\sigma\left(\mathbf{a}\right) d\mu\left(p\right) = 0,$$

that is, σ is a max-min solution for all mixed strategy σ over support $A_{\max\min}$.$\blacksquare$

Characterization of the welfare schedule $W_N(p)$ and the conditional expected social welfare function $W_N(p_0|a)$:

To derive the conditional expected social welfare function $W_N(p_0|a)$, we begin with the recursion for the welfare schedule $W_N(p)$ when the firm is faced with policy $\mathbf{a}$:

$$\begin{aligned}
W_N\left(p\right) &= -\alpha N k_N(p) dt - \gamma N\left[s + \phi\alpha(k_N(p) - z)\right] dt \\
&\quad + \lambda N k_N(p) p dt\left[CS + \Pi + \left(N-1\right)\theta\Pi\right] + (1 - \lambda N k_N(p) p dt) e^{-rdt} W_N\left(p + dp\right).
\end{aligned}$$

The welfare schedule can be transformed into the following differential equation:

$$\begin{aligned}
0 &= -\alpha N k_N\left(p\right) - \gamma N\left[s + \phi\alpha(k_N(p) - z)\right] + \lambda N k_N\left(p\right) p\left[CS + \left(1 + \left(N-1\right)\theta\right)\Pi\right] \\
&\quad - \left(r + \lambda N k_N\left(p\right) p\right) W_N\left(p\right) - \lambda N k_N\left(p\right) p\left(1-p\right) W_N'\left(p\right).
\end{aligned} \tag{36}$$

When $k_N\left(p\right) = 1$ (i.e., $p > q_N(\mathbf{a})$), the solution to this differential equation is

$$\begin{aligned}
W_N\left(p\right) &= -\frac{\alpha N}{r} - \frac{N\gamma\left[s + \alpha\phi\left(1-z\right)\right]}{r} \\
&\quad + \left\{\frac{\lambda N\left(r\left[CS + \left(1 + \left(N-1\right)\theta\right)\Pi\right] + N\alpha + N\gamma\left[s + \alpha\phi\left(1-z\right)\right]\right)}{r\left(r + \lambda N\right)}\right\} p + B_W^H p\left(\frac{1-p}{p}\right)^{\frac{r+\lambda N}{\lambda N}},
\end{aligned}$$

where B_W^H is a constant. When $k(p) = z$ (i.e., $p < p_N(\mathbf{a})$), the solution to this differential equation is

$$W_N(p) = -\frac{N\alpha z}{r} - \frac{N\gamma s}{r} + \frac{\lambda Nz\left(r\left[CS + (1 + (N-1)\theta)\Pi\right] + \alpha Nz + Ns\gamma\right)}{r(r + \lambda Nz)}p.$$

When $k_N(p) \in (z, 1)$, the differential equation (36) does not have a closed form solution.

The conditional expected social welfare function $W_N(p_0|a)$ is found by evaluating $W_N(p)$ at $p = p_0$.■

Proof of Proposition 8:

We employ the same logic used to prove Proposition 3: when $\gamma = 0$ it suffices to show that if we can induce $k_N(p) = k^*(p)$ through an appropriate choice of (z, s, ϕ), then that subsidy policy must indeed maximize $W_N(p_0|\mathbf{a})$. Now, recall that under the equilibrium policy there is no free riding if and only if

$$\theta \leq \overline{\theta}(z, s) = 1 - \frac{\Pi - \left(\frac{s - \alpha z}{r}\right)}{\Pi}\frac{r}{r + \lambda z}, .$$

When $z = 0$, $s = r\theta\Pi$, then it can be verified that $\overline{\theta}(z, s) = \theta$, which is just enough to eliminate the free-rider problem. The equilibrium investment policy in this case is

$$k_N(p) = \begin{cases} 1 & \text{if } p > p_N(0, r\theta\Pi, \phi) \\ 0 & \text{if } p \leq p_N(0, r\theta\Pi, \phi), \end{cases},$$

where

$$p_N(0, r\theta\Pi, \phi) = \frac{\alpha(1 - \phi)}{\lambda(1 - \theta)\Pi}.$$

By setting $\phi = \frac{CS + N\theta\Pi}{CS + \Pi + (N-1)\theta\Pi}$, we can make $p_N(0, r\theta\Pi, \phi) = p_N^*(\mathbf{a})$. Thus, the first-best investment policy can be induced by setting $z = 0$ and using a combination of unrestricted funding $s = r\theta\Pi$ and a matching rate $\phi = \frac{CS + N\theta\Pi}{CS + \Pi + (N-1)\theta\Pi}$.■

Proof of Proposition 9:

Suppose an *ex post* equilibrium exists. Then, in particular, that equilibrium would solve the optimization in (25) for the particular case of $p_0 = 1$. But when $p_0 = 1$, the solution to this optimization is $\mathbf{a} = (0, 0, 0)$. To see why formally, we note that from the derivation of $W_N(p_0|\mathbf{a})$ above when $p_0 = 1$, we have

$$
\begin{aligned}
W_N(1|\mathbf{a}) = {} & -\frac{\alpha N}{r} + \frac{\lambda N}{(r+\lambda N)}\left\{\frac{r\left[CS + (1+(N-1)\theta)\Pi\right] + N\alpha}{r}\right\} \\
& - \frac{N\gamma\left[s + \alpha\phi(1-z)\right]}{(r+\lambda N)}.
\end{aligned}
$$

This implies that the *ex post* equilibrium policy must be $\mathbf{a} = (0,0,0)$. This policy must, then, also solve the optimization in (25) if the government knew that $p_0 \in \left(p^{**}_{\gamma N}, p_N(0,0,0)\right)$. But in such a case the *ex post* equilibrium policy $(0,0,0)$ will not induce investment and would be inferior to policy $\left(0,0,\phi^{**}_{\gamma N}\right)$, which contradicts $\mathbf{a} = (0,0,0)$ being an *ex post* equilibrium. Since the supposition that an *ex post* equilibrium exists is the root cause of the contradiction, it must be false.■

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